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Martingale-observables for dipolar SLE with the Dirichlet boundary condition

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Martingale-observables for dipolar SLE with the Dirichlet boundary condition

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Abstract

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In this thesis, we study a version of dipolar conformal field theory based on the central charge modification of the Gaussian free field with the Dirichlet boundary condition. We prove that correlation functions of certain family of fields in this theory under the insertion of the boundary condition changing operator are martingale-observables for dipolar SLE. We adopt and modify definitions and basic properties developed in the study of a version of chordal conformal field theory to implement this version of dipolar conformal field theory. Using techniques in a dipolar conformal field theory, we prove the restriction property of dipolar SLE(8/3), Friedrich-Werner's formula in the dipolar case and the Cardy-Zhan observables.

Key words: Dipolar conformal field theory, Dipolar SLE, martingale-observables
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Contents

Chapter 1

Introduction

The Schramm-Loewner evolution (SLE) was proposed by Schramm [22] as a family of random planner curves that have been proven to be the scaling limit of critical two-dimensional lattice models in statistical physics. His approach has achieved the rigorous proof of important questions about the planner models in physics, in particular, some very non-trivial predictions of conformal field theory (CFT). For example, the serial papers [15, 16, 17, 18, 19] by Lawler, Schramm and Werner prove the conjecture of Mandelbrot that the boundary of planner Brownian motion has fractal dimension 4/3, and Smirnov [23] proved the Cardy's formula, which is the limit of crossing probability of the critical percolation model on a triangular lattice. Motivated by results about these connections, there are many papers for understanding the precise relation between SLE and CFT, (see [8] and the physics literature [2, 3, 6]). CFT has also had a marked impact in algebra and geometry. However, there are still many non-rigorous arguments such as the renormalization group.

In [12], Kang and Makarov introduced CFT on the view of random or statistical fields. More precisely, they described a version of CFT of statistical fields generated by non-random modification of the Gaussian free field (GFF). Consequently, they gave mathematical definitions and proofs about the theory developed by Belavin, Polyakov, and Zamolodchikov [4] such as an operator algebra formalism of Virasoro algebra, Ward identity, stress-energy tensor, vertex fields, and the BPZ equation, which is a linear differential equation in terms of correlation functions of random fields constructed from GFF under OPE multiplications. Also, it is well-known in physical literature (see e.g. [6]) that, under the insertion of the boundary condition changing operator, the correlation functions of a certain class of fields are martingale-observables for chordal SLE_κ. The boundary condition changing operator is defined as a normalized rooted vertex field with the boundary dimension $h_{1,2} = \frac{6-\kappa}{2\kappa}$ at $p, q \in \partial D$, where p is the start point and q is the end point of SLE_κ in a simply connected domain D.

There have been two types of SLE thoroughly studied: the chordal and the radial. Chordal SLE describes random curves joining two boundary points on the boundary of a simply connected domain, and radial SLE describes random curves joining one boundary point and one interior point of a simply connected domain. Dipolar (or strip) is another type of SLE, which was introduced by Zhan in [25] or by Bauer, Bernard and Houdayer in the physics literature [1]. Dipolar SLE expresses random curves joining one marked boundary point p and a connected marked boundary arc Q of a simply connected domain where Q does not include p.

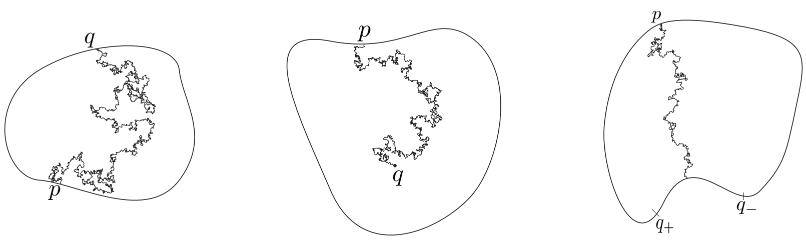

Figure 1.1: Chordal, radial and dipolar SLEs.

In this thesis, we study a version of dipolar conformal field theory with the Dirichlet boundary condition in a simply connected domain with two marked boundary points q_-, q_+, which are two endpoints of Q. To implement the dipolar conformal field theory, we first review definitions and basic properties developed in the study of the chordal case [12] and the radial case [11]. However, we introduce a different version of modification of GFF so that the corresponding CFT is connected with the dipolar SLE_κ. In the physics literature [1], inserting of boundary operators $\psi_{1;2}(p)$ $(p \in \partial D)$ and $\psi_{0;1/2}(q_\pm)$, all correlation functions of the fields in a certain collection are martingale-observables for dipolar SLE$_\kappa$ from p to arc Q with endpoints $q_\pm$. We present its proof after we give a precise definition for $\Psi(p;q_-,q_+)(\equiv \psi_{1;2}(p)\psi_{0;1/2}(q_-)\psi_{0;1/2}(q_+)$ up to boundary puncture operators) as a boundary vertex field rooted at $q_\pm$ of a single variable p. As applications, we introduce the restriction property of the dipolar SLE$_{8/3}$ and the Cardy-Zhan observables. We also explain the similarities and differences of the chordal, the radial, and the dipolar conformal field theory. For example, the neutrality condition is required for the rooted vertex fields to be well-defined Fock space fields in all theories. However, each theory has different background charge in terms of Coulomb gas formalism.

Chapter 2

Main results

2.1 Dipolar SLE martingale-observables

Let us consider a simply connected domain (D, p, Q) with a marked boundary point p and a marked boundary arc $Q \subseteq \partial D$ such that $p \notin \bar{Q}$. We denote by q_-, q_+ two endpoints of Q such that q_-, p, q_+ are positively oriented. A dipolar Schramm-Loewner evolution (SLE_κ) in (D, p, Q) with a parameter κ ($\kappa > 0$) is the conformally invariant law on random curves from the point p to the arc Q described by the solution $\psi_t(z)$ of the dipolar Loewner equation

$$\partial_t \psi_t(z) = \coth_2(\psi_t(z) - \xi_t), \quad (\xi_t = \sqrt{\kappa} B_t), \tag{2.1.1}$$

where $\coth_2(z) := \coth(\frac{1}{2}z)$ and B_t is a one-dimensional standard Brownian motion with $B_0 = 0$. As an initial data, $\psi_0 : (D, p, Q) \to (\mathbb{S}, 0, \mathbb{R} + \pi i)$ is the conformal map from D onto the strip $\mathbb{S} := \{z \in \mathbb{C} \,|\, 0 < \operatorname{Im} z < \pi\}$. Let τ_z be the supremum of the set of t such that $\psi_t(z)$ is well-defined. Then for all t,

$$w_t : (D_t, \gamma_t, Q) \to (\mathbb{S}, 0, \mathbb{R} + \pi i), \qquad w_t(z) := \psi_t(z) - \xi_t$$

is a conformal map from

$$D_t := \{z \in D : \tau_z > t\}$$

onto the strip $\mathbb{S}$. The *dipolar SLE curve* γ is defined by

$$\gamma_t \equiv \gamma(t) := \lim_{z \to 0} w_t^{-1}(z).$$

It is well known ([24, 26]) that the SLE trace exists in the dipolar case.

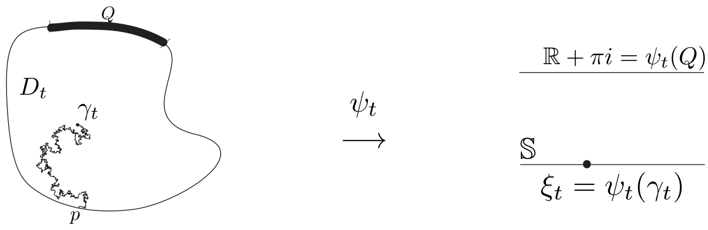

Figure 2.1: Dipolar SLE curve and hull.

To define dipolar SLE martingale-observables, let us recall the definition of non-random conformal fields. See [12, Lecture 4.1] for more details. A non-random *conformal* field M is an assignment of a (smooth) function $(M \,\|\, \phi): \ \phi U \to \mathbb{C}$ to each local chart $\phi : U \to \phi U$. A non-random conformal Fock space field M is a $[\lambda, \lambda_*]$-*differential* if for any two overlapping charts $\phi, \widetilde{\phi}$, we have

$$(M \,\|\, \phi) = (h')^{\lambda} (\overline{h'})^{\lambda_*} (M \,\|\, \widetilde{\phi}) \circ h,$$

where $h = \widetilde{\phi} \circ \phi^{-1}: \ \phi(U \cap \widetilde{U}) \to \widetilde{\phi}(U \cap \widetilde{U})$ is the transition map. A pair $[\lambda, \lambda_*]$ is called degrees or conformal dimensions of M. Non-random conformal Fock space fields M are called *pre-pre-Schwarzian forms*, *pre-Schwarzian forms*, and *Schwarzian forms* of order $\mu (\in \mathbb{C})$ if the following transformation laws hold:

$$(M \,\|\, \phi) = (M \,\|\, \widetilde{\phi}) \circ h + \mu \log h', \ (M \,\|\, \phi) = h' \,(M \,\|\, \widetilde{\phi}) \circ h + \mu \frac{h''}{h'},$$

and

$$(M \,\|\, \phi) = (h')^2 \,(M \,\|\, \widetilde{\phi}) \circ h + \mu S_h,$$

respectively. Here, S_h is the Schwarzian derivative of h, $S_h = (h''/h')' - \frac{1}{2}(h''/h')^2$.

A non-random (conformal) field M of n variables in D is said to be a *martingale-observable* for dipolar SLE$_\kappa$ if for any $z_1, \cdots, z_n \in D$, the process

$$M_t(z_1, \cdots, z_n) = (M_{D_t, \gamma_t, Q} \,\|\, \mathrm{id})(z_1, \cdots, z_n) = (M \,\|\, w_t^{-1})(z_1, \cdots, z_n)$$

is a local martingale on dipolar SLE probability space. (The process M_t is stopped when any z_j exits D_t.) For example, we can use the identity chart of $\mathbb{S}$. Then for $[h, h_*]$-differentials M with boundary conformal dimensions

$h_\pm$ at $q_\pm$, we have

$$M_t(z) = (w_t'(z))^h (\overline{w_t'(z)})^{h_*} (w_t'(q_-))^{h_-} (w_t'(q_+))^{h_+} M(w_t(z)).$$

If M is a pre-Schwarzian form of order μ, then

$$M_t(z) = w_t'(z) M(w_t(z)) + \mu \frac{w_t''(z)}{w_t'(z)}.$$

Similarly, for a Schwarzian form M of order μ, we have

$$M_t(z) = (w_t'(z))^2 M(w_t(z)) + \mu S_{w_t}(z).$$

2.2 A dipolar CFT

All our fields in this thesis are Fock space (correlational) fields constructed from the Gaussian free field $\Phi_{(0)}$ with the Dirichlet boundary condition, its derivatives, and Wick's exponentials $e^{\odot \alpha \Phi_{(0)}}$ ($\alpha \in \mathbb{C}$) by means of Wick's calculus. (An alternate notation for Wick's exponentials of $\Phi_{(0)}$ is $:e^{\alpha \Phi_{(0)}}:$.) For Fock space fields $X_1, \cdots, X_n$ and distinct points (*nodes*) $z_1, \cdots, z_n$ in D, a *correlation function*

$$\mathbf{E}[X_1(z_1) \cdots X_n(z_n)]$$

is defined by Wick's formula. We will review its definition and basic properties in Section 3.3. For example, we define

$$\mathbf{E}[\Phi_{(0)}(z)] = 0, \qquad \mathbf{E}[\Phi_{(0)}(z_1)\Phi_{(0)}(z_2)] = 2G(z_1, z_2)$$

(G is the Green's function for D with the Dirichlet boundary condition) and

$$\mathbf{E}[\Phi_{(0)}(z_1) \cdots \Phi_{(0)}(z_n)] = \sum \prod_k \mathbf{E}[\Phi_{(0)}(z_{i_k})\Phi_{(0)}(z_{j_k})],$$

where the sum is over all partitions of the set $\{1, \cdots, n\}$ into disjoint pairs $\{i_k, j_k\}$.

For a simply connected domain D with a marked boundary arc $Q \subseteq \partial D$, we consider a conformal map

$$w \equiv w_{D,Q} : (D, Q) \to (\mathbb{S}, \mathbb{R} + \pi i),$$

from D onto the strip $\mathbb{S}$. For a fixed parameter $b \in \mathbb{R}$, we define central charge modifications $\Phi \equiv \Phi_{(b)}$ of the Gaussian free field $\Phi_{(0)}$ by

$$\Phi_{(b)} = \Phi_{(0)} - 2b \arg w'$$

(cf. central charge modifications of the Gaussian free field in the chordal case, see e.g., [12, Lecture 10.1]). We denote by $\mathcal{F}_{(b)}$ the operator product expansion (OPE) family of $\Phi_{(b)}$, the algebra over $\mathbb{C}$ spanned by $1, \partial^j \bar{\partial}^k \Phi_{(b)}$, and derivatives of the vertex fields $\partial^j \bar{\partial}^k \mathcal{V}^{\alpha}_{(b)}$ $(\alpha \in \mathbb{C})$ under OPE multiplications $*$. We will recall the definition of operator product expansion and its basic properties in Section 3.4. As in the chordal case ([12, Lectures 3.3, 10.2]), vertex fields are defined as OPE exponentials of the bosonic field $\Phi = \Phi_{(b)}$:

$$\mathcal{V}^{\alpha} \equiv \mathcal{V}^{\alpha}_{(b)} = e^{*\alpha\Phi} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \Phi^{*n}.$$

As in the radial case ([11]), we extend the OPE family $\mathcal{F}_{(b)}$ to include the bi-variant chiral bosonic field $\Phi^+_{(b)}(z, z_0)$, its conjugate, and (derivatives of) chiral multi-vertex fields

$$\mathcal{O}^{(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*)}(\boldsymbol{z}), \quad (\boldsymbol{\sigma} = (\sigma_1 \cdots, \sigma_n), \, \boldsymbol{\sigma}_* = (\sigma_{*1} \cdots, \sigma_{*n}), \, \boldsymbol{z} = (z_1, \cdots, z_n))$$

(its precise definition is given in Section 5.2) with the *neutrality condition*

$$\sum_{j=1}^{n} (\sigma_j + \sigma_{*j}) = 0. \tag{2.2.1}$$

As a multivalued field, the bi-variant chiral bosonic field $\Phi^+_{(b)}(z, z_0)$ can be expressed in terms of 1-point formal field $\Phi^+_{(b)}$ as follows:

$$\Phi^+_{(b)}(z, z_0) = \Phi^+_{(b)}(z) - \Phi^+_{(b)}(z_0), \quad \Phi^+_{(b)}(z) = \Phi^+_{(0)}(z) + ib \log \frac{w'(z)}{1 - w(z)^2},$$

where w is a conformal map from (D, q_-, q_+) onto $(\mathbb{H}, -1, 1)$ and the formal 1-point field $\Phi^+_{(0)}$ (which can be interpreted as a "holomorphic" part of $\Phi_{(0)}$

in the sense that $\Phi_{(0)}(z) = 2\,\mathrm{Re}\,\Phi^+_{(0)}(z))$ has formal correlations

$$\mathbf{E}[\Phi^+_{(0)}(z)\Phi^+_{(0)}(z_0)] = \log\frac{1}{w(z)-w(z_0)},$$
$$\mathbf{E}[\Phi^+_{(0)}(z)\overline{\Phi^+_{(0)}(z_0)}] = \log(w(z)-\overline{w(z_0)}).$$

The chiral multi-vertex field $\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]$ can be interpreted as the OPE exponential of the formal field

$$i\sum \sigma_j\Phi^+_{(b)}(z_j) - \sigma_{*j}\Phi^-_{(b)}(z_j),$$

where $\Phi^-_{(b)} = \overline{\Phi^+_{(b)}}$. It turns out that the formal fields $\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]$ are well-defined Fock space fields if and only if the neutrality condition holds. Under the neutrality condition, multi-vertex fields $\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]$ are $\mathrm{Aut}(D,q_\pm)$-invariant primary fields. See Section 3.5 for the definition of conformal invariance.

In the chordal (radial) case, under the insertion of Wick's exponential

$$e^{\odot ia\Phi^+_{(0)}(p,q)}\,(p,q\in\partial D, p\neq q), \qquad \Big(e^{\odot -a\,\mathrm{Im}\,\Phi^+_{(0)}(p,q)}\,(p\in\partial D, q\in D)\Big),$$

with $a=\sqrt{2/\kappa}$ and $b = a(\kappa/4-1)$, all fields in the extended OPE family $\mathcal{F}_{(b)}$ of $\Phi_{(b)}$ satisfy the "field Markov property" with respect to chordal (radial) SLE filtration, respectively. (See [2, 21] for the chordal case and [3, 5] for the radial case from the physics perspective, cf. [12, Proposition 14.3] and [11, Theorem 1.1].) The dipolar version of this theorem can be stated as follows. (As we mentioned in Chapter 1, special cases of the following theorem are well known in physics literature, e.g., [1].)

Theorem 2.2.1. *For the tensor product $X = X_1(z_1)\cdots X_n(z_n)$ of fields X_j in the OPE family $\mathcal{F}_{(b)}$ of $\Phi_{(b)}$, the non-random fields*

$$\mathbf{E}\,[e^{\odot\frac12 ia(\Phi^+_{(0)}(p,q_-)+\Phi^+_{(0)}(p,q_+))}X] \qquad (a=\sqrt{2/\kappa},\, b=\sqrt{\kappa/8}-\sqrt{2/\kappa}) \quad (2.2.2)$$

are martingale-observables for dipolar SLE_κ.

2.3 Boundary condition changing operators and rooted vertex fields

The boundary condition changing operator on Fock space functionals/fields is the insertion of

$$e^{\odot \frac{1}{2} ia(\Phi^+_{(0)}(p,q_-)+\Phi^+_{(0)}(p,q_+))}. \tag{2.3.1}$$

While this field does not belong to the extended OPE family $\mathcal{F}_{(b)}$, we further extend the OPE family to contain the multi-vertex fields $\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+]$ rooted at two points q_-, q_+ with the neutrality condition $(\sigma_- + \sigma_+ + \sum_{j=1}^n (\sigma_j + \sigma_{*j}) = 0)$ so that the field (2.3.1) is represented as

$$\frac{\Psi(p)}{\mathbf{E}\,[\Psi(p)]}, \qquad \Psi \equiv \Psi(\cdot; q_-, q_+) := \mathcal{O}[a, 0; -\frac{1}{2}a, -\frac{1}{2}a] \in \mathcal{F}_{(b)}.$$

Similarly as in the radial case, the definition of rooted multi-vertex fields can be obtained by normalizing multi-vertex fields

$$\mathcal{O}^{(\boldsymbol{\sigma}, \boldsymbol{\sigma_*})} \mathcal{O}^{(\sigma_-)}(\eta_-) \mathcal{O}^{(\sigma_+)}(\eta_+)$$

and taking a limit as (η_-, η_+) approaches (q_-, q_+). This rooting procedure also gives rise to the definition of the *normalized tensor product* of rooted vertex fields as

$$\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+] \star \mathcal{O}[\boldsymbol{\tau}, \boldsymbol{\tau_*}; \tau_-, \tau_+] = \mathcal{O}[\boldsymbol{\sigma} + \boldsymbol{\tau}, \boldsymbol{\sigma_*} + \boldsymbol{\tau_*}; \sigma_- + \tau_-, \sigma_+ + \tau_+],$$

where $\boldsymbol{\sigma}, \boldsymbol{\sigma_*}, \boldsymbol{\tau}, \boldsymbol{\tau_*}$ are divisors, maps from D to $\mathbb{R}$ which take the value 0 at all but finitely many points. Like the multi-vertex fields, the rooted multi-vertex field $\mathcal{O}^{(\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+)}(\boldsymbol{z})$ can be interpreted as the OPE exponential of the field

$$i\sigma_- \Phi^+_{(b)}(q_-) + i\sigma_+ \Phi^+_{(b)}(q_+) + i \sum \sigma_j \Phi^+_{(b)}(z_j) - \sigma_{*j} \Phi^-_{(b)}(z_j), \tag{2.3.2}$$

where $\Phi^+_{(b)}(q_\pm) = \Phi^+_{(0)}(q_\pm)$. In the above expression, we do not need the terms $\Phi^-_{(0)}(q_\pm)$ since $\Phi^+_{(0)}(q_\pm)$ and $\Phi^-_{(0)}(q_\pm)$ are not linearly independent. Indeed, $\Phi^-_{(0)}(q_\pm) = -\Phi^+_{(0)}(q_\pm)$ because $\Phi_{(0)}(q_\pm) = 0$. Under the neutrality condition, rooted vertex fields $\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+]$ are $\mathrm{Aut}(D, q_\pm)$-invariant primary fields with conformal dimensions $[\boldsymbol{h}, \boldsymbol{h}_*; h_-, h_+]$:

$$h_j = \frac{\sigma_j^2}{2} - \sigma_j b, \quad h_{*j} = \frac{\sigma_{*j}^2}{2} - \sigma_{*j} b, \quad h_\pm = \frac{\sigma_\pm^2}{2}.$$

We extend $\mathcal{F}_{(b)}$ by adding the generators, (2.3.2) and the rooted vertex fields with the neutrality conditions. We call this extended collection of fields the *extended OPE family* of $\Phi_{(b)}$. For a rooted vertex field $\mathcal{O} \equiv \mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+]$ with the neutrality condition, $\mathbf{E}\,[\Psi(p)\mathcal{O}]$ is not well-defined because both $\Psi(p) \equiv \Psi(p; q_-, q_+)$ and $\mathcal{O}$ have nodes at $q_\pm$. However, Theorem 2.2.1 can be modified for a rooted multi-vertex field with the neutrality condition (cf. [11, Theorem 1.2] for its radial version).

Theorem 2.3.1. *For a rooted multi-vertex field $\mathcal{O} \equiv \mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+]$ with the neutrality condition, the non-random field*

$$\frac{\mathbf{E}[\Psi(p) \star \mathcal{O}]}{\mathbf{E}[\Psi(p)]}$$

is a dipolar SLE_κ *martingale-observable.*

Theorems 2.2.1 – 2.3.1 can be extended to the fields in the extended OPE family of $\Phi_{(b)}$.

2.4 Examples of martingale-observables

We use conformal field theory to present the proof for the restriction property of dipolar $\mathrm{SLE}_{8/3}$: for a fixed compact hull K (i.e., K is a compact set such that $\mathbb{S}\setminus K$ is a simply connected subdomain of $\mathbb{S}$) with $K \cap (\mathbb{R}+\pi i) = \emptyset$, the dipolar $\mathrm{SLE}_{8/3}$ path in $(\mathbb{S}, 0, -\infty, \infty)$ conditioned to avoid K has the same distribution as the dipolar $\mathrm{SLE}_{8/3}$ path in $(\mathbb{S} \setminus K, 0, -\infty, \infty)$. It is equivalent to Theorem 2.4.1 below. To state it, let us recall the definition of the strip capacity (e.g., see [25]) of a compact hull K. For a compact hull K with $K \cap (\mathbb{R} + \pi i) = \emptyset$, there is a unique conformal transformation from $(\mathbb{S} \setminus K, -\infty, \infty)$ onto $(\mathbb{S}, -\infty, \infty)$ such that

$$\lim_{z \to \pm\infty} \psi_K(z) - z = \pm\, s$$

for some $s \geq 0$. Here, $z \to \pm\infty$ means that $z \in \mathbb{S}$ and $\operatorname{Re} z \to \pm\infty$. This s is called the strip capacity of K and denoted by $\operatorname{scap}(K)$.

Theorem 2.4.1. *For a given compact hull K,*

$$\mathbb{P}(\mathrm{SLE}_{8/3} \textit{ path avoids } K) = \psi_K'(0)^\lambda e^{-2\mu\,\mathrm{scap}(K)}, \quad (\lambda = 5/8,\, \mu = 5/96). \tag{2.4.1}$$

In the half-plane uniformization, (2.4.1) reads as

$$\mathbb{P}(\mathrm{SLE}_{8/3} \text{ path avoids } K) = \psi_K'(0)^\lambda (\psi_K'(-1)\psi_K'(1))^\mu, \tag{2.4.2}$$

where K is a fixed compact hull with $\partial K \cap \mathbb{R} \subseteq (-1,1) \setminus \{0\}$ and ψ_K is the conformal transformation from $(\mathbb{H} \setminus K, -1, 1)$ onto $(\mathbb{H}, -1, 1)$ such that $\psi_K'(-1) = \psi_K'(1)$. Compare (2.4.2) to the restriction property of chordal $\mathrm{SLE}_{8/3}$ and radial $\mathrm{SLE}_{8/3}$, see [20, Theorem 6.1], [14, Theorem 6.26]. The restriction exponents λ and μ can be explained in term of conformal dimensions of Ψ. Let us introduce the *effective boundary condition changing operator* Ψ^{eff} :

$$\Psi^{\mathrm{eff}} := \Psi\, \mathcal{P}_{(b)}(q_\pm), \tag{2.4.3}$$

where $\mathcal{P}_{(b)}(q_\pm)$ is the "boundary puncture operator" defined as a $-\frac{1}{2}b^2$-boundary differential at $q_\pm$ and $\mathcal{P}(q_\pm) \equiv 1$ in the identity chart of $\mathbb{H}$. Then

$$\lambda = h(\Psi) := \frac{a^2}{2} - ab = \frac{6-\kappa}{2\kappa}, \qquad \mu = h_\pm(\Psi^{\mathrm{eff}}) := \frac{a^2}{8} - \frac{b^2}{2} = \frac{(\kappa-2)(6-\kappa)}{16\kappa}.$$

As an application of the restriction property of dipolar $\mathrm{SLE}_{8/3}$, we prove a dipolar version of Friedrich-Werner formula.

Theorem 2.4.2. *For distinct $x_j \in \mathbb{R}\setminus\{0\} (j = 1, \cdots, n)$ and for $b = -\sqrt{3}/6$,*

$$\lim_{\varepsilon \to 0} \varepsilon^{-2n} \mathbb{P}(\mathrm{SLE}_{8/3} \textit{ hits all slits } [x_j, x_j + i\varepsilon\sqrt{2}]) = \widehat{\mathbf{E}}[T(x_1) \cdots T(x_n) \,\|\, \mathrm{id}_{\mathbb{S}}].$$

Compare Theorem 2.4.2 to its chordal and radial version (see [8, Proposition 1] and [11, Theorem 4.4], respectively).

In the last section we identify the probability that $z (\in D)$ is swallowed by dipolar $\mathrm{SLE}_\kappa (\kappa > 4)$ hulls and the probability that z is to the left (right) of dipolar SLE_κ paths with correlations of primary observables (Cardy-Zhan observables) by the method of "screening."

Chapter 3

Basic properties of conformal Fock space fields

Here, we include some definitions and concepts of conformal Fock space fields developed in [12] so that this thesis can be read as a self-contained article.

3.1 Gaussian free field on simply connected domain

Let D be a simply connected domain and G_D be the Green's function on D with *Dirichlet boundary condition*. For example, in the upper half-plane $\mathbb{H} = \{z : 0 < \mathrm{Im} z\}$, we have

$$G_{\mathbb{H}}(\zeta, z) = \log \left| \frac{\zeta - \bar{z}}{\zeta - z} \right|,$$

and in the strip domain $\mathbb{S} = \{z : 0 < \mathrm{Im} z < \pi\}$, we have

$$G_{\mathbb{S}}(\zeta, z) = \log \left| \frac{\sinh_2(\zeta - \bar{z})}{\sinh_2(\zeta - z)} \right|,$$

where $\sinh_2(\cdot) = \sinh(\cdot/2)$. *The Gaussian free field* $\Phi \equiv \Phi_{(0)}$ on D with Dirichlet boundary condition is a (real, centered) Gaussian field indexed by the Dirichlet energy space $\mathcal{E}(D)$. The Hilbert space $\mathcal{E}(D)$ can be defined as the completion of test functions $f \in C_0^\infty(D)$ with respect to the norm

$$\|f\|_{\mathcal{E}}^2 = \iint 2G_D(\zeta, z) f(\zeta)\overline{f(z)} dA(\zeta) dA(z),$$

where A is the area measure. It means that $\Phi : \mathcal{E}(D) \to L^2(\Omega, \mathbf{P})$ is an isometry such that $\Phi(f)$ is a (real, centered) Gaussian random variable for each $f \in \mathcal{E}(D)$. By definition, for n distinct points $z_j \in D$, the n-point correlation function of $\mathbf{E}[\Phi(z_1)\cdots\Phi(z_n)]$ is the unique continuous function such that

$$\mathbf{E}[\Phi(f_1)\cdots\Phi(f_n)] = \int f_1(z_1)\cdots f_n(z_n)\mathbf{E}[\Phi(z_1)\cdots\Phi(z_n)]dA(z_1)\cdots dA(z_n)$$

for all test functions f_j. It is clear that the 2-point function is $2G_D(z, w)$ and it can be shown that

$$\mathbf{E}[\Phi(z_1)\cdots\Phi(z_n)] = \sum \prod_k 2G_D(z_{i_k}, z_{j_k}),$$

where the sum is over all partitions of the set $\{1, \cdots, n\}$ into disjoint pairs $\{i_k, j_k\}$. The fields $J = \partial\Phi$, $\overline{J} = \overline{\partial}\Phi$, and higher order derivatives are well-defined as Gaussian distributional fields, e.g.,

$$J(f) = -\Phi(\partial f), \quad f \in C_0^\infty(D),$$

so J is a map $\mathcal{E}(D) \to L^2(\Omega, \mathbf{P})$. We can compute the correlation functions of the derivatives by differentiating the correlation functions of the Gaussian free field. For example, for $\zeta \neq z$, we have:

$$\mathbf{E}[J(\zeta)\Phi(z)] = 2\partial_\zeta G_D(\zeta, z),$$
$$\mathbf{E}[J(\zeta)J(z)] = 2\partial_\zeta\partial_z G_D(\zeta, z).$$

3.2 Fock space correlation functionals

This section is borrowed from [12, Lecture 1.3]. By definition, *basic* correlation functionals are formal expressions of the type (Wick's product of $X_j(z_j)$)

$$\mathcal{X} = X_1(z_1) \odot \cdots \odot X_n(z_n),$$

where points $z_j \in D$ are not necessarily distinct and X_j's are derivatives of the Gaussian free field (i.e., $X_j = \partial^j \bar{\partial}^k \Phi$), or Wick's exponentials

$$e^{\odot \alpha \Phi} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \Phi^{\odot n}.$$

The constant 1 is also included to the list of basic functionals. We write $S_{\mathcal{X}}$ for the set of all points z_j (the *nodes* of $\mathcal{X}$) in the expression of $\mathcal{X}$.

For derivatives X_{jk} of the Gaussian free field and basic functionals of the form

$$\mathcal{X}_j = X_{j1}(z_{j1}) \odot \cdots \odot X_{jn_j}(z_{jn_j}),$$

we define the tensor product $\mathcal{X}_1 \cdots \mathcal{X}_m$ by

$$\mathcal{X}_1 \cdots \mathcal{X}_m = \sum \prod_{\{v,v'\}} \mathbf{E}[X_v(z_v) X_{v'}(z_{v'})] \bigodot_{v''} X_{v''}(z_{v''}), \tag{3.2.1}$$

where the sum is taken over all Feynman diagrams with vertices v labeled by functionals X_{jk} such that there are no contractions of vertices with the same j, and the Wick's product is taken over unpaired vertices v''. By definition, $\mathbf{E}[X_v(z_v) X_{v'}(z_{v'})]$ in (3.2.1) are given by the 2-point functions of derivatives of the Gaussian free field, e.g.,

$$\mathbf{E}[\partial^j \Phi(\zeta) \partial^k \Phi(z)] = \partial_\zeta^j \partial_z^k \mathbf{E}[\Phi(\zeta)\Phi(z)] = 2\partial_\zeta^j \partial_z^k G(\zeta, z).$$

For example, the Feynman diagram with two edges $\{1,4\}, \{3,5\}$ and two unpaired vertices $2, 6$ corresponds to

$$\begin{aligned} &(\Phi(z_1) \odot \Phi(z_2) \odot \Phi(z_3))(\Phi(z_4) \odot \Phi(z_5) \odot \Phi(z_6)) \\ &:= \mathbf{E}[\Phi(z_1)\Phi(z_4)]\mathbf{E}[\Phi(z_3)\Phi(z_5)]\Phi(z_2) \odot \Phi(z_6). \end{aligned}$$

The definition of tensor product can be extended to general correlation functionals by linearity. The tensor product of correlation functionals is commutative and associative, see [12, Proposition 1.1].

We define the correlation function $\mathbf{E}[\mathcal{X}]$ of $\mathcal{X}$ by linearity, $\mathbf{E}[1] = 1$, and

$$\mathbf{E}[X_1(z_1) \odot \cdots \odot X_n(z_n)] = 0,$$

where X_j are derivatives of Φ. For example, $\mathbf{E}[e^{\odot\alpha\Phi(z)}] = 1$ and

$$\mathbf{E}[\Phi(z_1)\cdots\Phi(z_n)] = \sum\prod_k 2G(z_{i_k}, z_{j_k}),$$

where the sum is over all partitions of the set $\{1, \cdots, n\}$ into disjoint pairs $\{i_k, j_k\}$.

If $\mathbf{E}[\mathcal{X}_1\mathcal{Y}] = \mathbf{E}[\mathcal{X}_2\mathcal{Y}]$ holds for all $\mathcal{Y}$ with nodes outside $S_{\mathcal{X}_1} \cup S_{\mathcal{X}_2}$, we identify $\mathcal{X}_1$ with $\mathcal{X}_2$ and write $\mathcal{X}_1 \approx \mathcal{X}_2$. We consider Fock space functionals modulo an ideal $\mathcal{N} = \{\mathcal{X} \approx 0\}$ of Wick's algebra. The concept of a correlation functional $\mathcal{X}$ can be extended to the case when some of the nodes of $\mathcal{X}$ lie on the boundary. For example, $e^{\odot\alpha\Phi(z)} = 1$ for $z \in \partial D$. The complex conjugation $\overline{\mathcal{X}}$ of $\mathcal{X}$ is defined (modulo $\mathcal{N}$) by the equation $\mathbf{E}[\overline{\mathcal{X}}\mathcal{Y}] = \overline{\mathbf{E}[\mathcal{X}\mathcal{Y}]}$ for all $\mathcal{Y}$'s of the form $\Phi(z_1)\odot\cdots\odot\Phi(z_n)$. For example, if $J = \partial\Phi$ in the half-plane $\mathbb{H}$ and if $z \in \partial\mathbb{H}$, then $J(z)$ is purely imaginary, i.e., $\overline{J(z)} = -J(z)$, and $J(z) \odot J(z)$ is real.

3.3 Fock space fields

This section is borrowed from [12, Lecture 1.4]. Basic Fock space fields X_α are formal expressions written as Wick's products of derivatives of the Gaussian free field Φ and Wick's exponential $e^{\odot\alpha\Phi}$, e.g., 1, $\Phi \odot \Phi \odot \Phi$, $\partial^2\Phi \odot \bar\partial\Phi$, $\partial\Phi \odot e^{\odot\alpha\Phi}$, etc. A general Fock space field is a linear combination of basic fields X_α,

$$X = \sum_\alpha f_\alpha X_\alpha,$$

where f_α's are arbitrary (smooth) functions in D. If $X_1, \cdots, X_n$ are Fock space fields and $z_1, \cdots, z_n$ are distinct points in D, then $\mathcal{X} = X_1(z_1)\cdots X_n(z_n)$ is a correlation functional.

We define the differential operators ∂ and $\bar\partial$ on Fock space fields by specifying their action on basic fields so that the action on Φ is consistent with the definition of $\partial\Phi, \bar\partial\Phi$ and so that

$$\partial(X \odot Y) = (\partial X) \odot Y + X \odot (\partial Y), \qquad \bar\partial(X \odot Y) = (\bar\partial X) \odot Y + X \odot (\bar\partial Y).$$

We extend this action to general Fock space fields by linearity and by Leibniz's rule with respect to multiplication by smooth functions. Then (modulo

$\mathcal{N}$)

$$\mathbf{E}[(\partial X)(z)\mathcal{Y}] = \partial_z \mathbf{E}[X(z)\mathcal{Y}], \qquad (z \notin S_{\mathcal{Y}}),$$

for all correlation functionals $\mathcal{Y}$.

By definition, X is *holomorphic* in D if $\bar{\partial} X \approx 0$, i.e., all correlation functions $\mathbf{E}[X(\zeta)\mathcal{Y}]$ are holomorphic in $\zeta \in D \setminus S_{\mathcal{Y}}$. For example, $J = \partial\Phi$, $X = J \odot J$ are holomorphic fields.

3.4 Operator product expansion

This section is borrowed from [12, Lectures 3.1 – 3.2]. Operator product expansion (OPE) is the expansion of the tensor product of two fields near diagonal. For example,

$$\Phi(\zeta)\Phi(z) = \log\frac{1}{|\zeta - z|^2} + 2c(z) + \Phi^{\odot 2}(z) + o(1) \qquad \text{as } \zeta \to z,\ \zeta \neq z, \tag{3.4.1}$$

where $c = \log C$ is the logarithm of conformal radius C, i.e., $c(z) = u(z, z)$, $u(\zeta, z) = G(\zeta, z) + \log|\zeta - z|$. The meaning of the convergence is the following: the equation

$$\mathbf{E}[\Phi(\zeta)\Phi(z)\mathcal{X}] = \log\frac{1}{|\zeta - z|^2}\mathbf{E}[\mathcal{X}] + 2c(z)\mathbf{E}[\mathcal{X}] + \mathbf{E}[\Phi^{\odot 2}(z)\mathcal{X}] + o(1)$$

holds for all Fock space correlation functionals $\mathcal{X}$ in D satisfying $z \notin S_{\mathcal{X}}$. To derive (3.4.1) we use Wick's formula (3.2.1),

$$\Phi(\zeta)\Phi(z) = \mathbf{E}[\Phi(\zeta)\Phi(z)] + \Phi(\zeta) \odot \Phi(z)$$

and the relation

$$\mathbf{E}[\Phi(\zeta)\Phi(z)] = 2G(\zeta, z) = \log\frac{1}{|\zeta - z|^2} + 2c(z) + o(1).$$

The convergence of $\Phi(\zeta) \odot \Phi(z)$ to $\Phi^{\odot 2}(z)$ means (by definition) that

$$\mathbf{E}[(\Phi(\zeta) \odot \Phi(z))\mathcal{X}] \to \mathbf{E}[\Phi^{\odot 2}(z)\mathcal{X}]$$

for every $\mathcal{X}$ such that $z \notin S_{\mathcal{X}}$.

If the field X is *holomorphic* (i.e., all correlation functions $\mathbf{E}[X(\zeta)\mathcal{Y}]$ are holomorphic in $\zeta \in D \setminus S_{\mathcal{Y}}$), then the operator product expansion is then defined as a (formal) Laurent series expansion

$$X(\zeta)Y(z) = \sum C_n(z)(\zeta - z)^n, \qquad \zeta \to z. \tag{3.4.2}$$

Since the function $\zeta \mapsto \mathbf{E}\, X(\zeta)Y(z)\mathcal{Z}$ is holomorphic in a punctured neighborhood of z, it has a Laurent series expansion.

There are only finitely many terms in the principle (or *singular*) part of the Laurent series (3.4.2). We use the notation $\sim$ for the singular part of the operator product expansion,

$$X(\zeta)Y(z) \sim \sum_{n<0} C_n(z)(\zeta - z)^n.$$

We also write $\operatorname{Sing}_{\zeta \to z} X(\zeta)Y(z)$ for $\sum_{n<0} C_n(z)(\zeta - z)^n$. It is clear that we can differentiate operator product expansions (3.4.2) both in ζ and z; and the differentiation preserves singular parts. For example,

$$J(\zeta)\Phi(z) \sim -\frac{1}{\zeta - z}, \qquad J(\zeta)J(z) \sim -\frac{1}{(\zeta - z)^2}.$$

The coefficients in the operator product expansions (e.g., $2c(z) + \Phi^{\odot 2}(z)$ in (3.4.1), $C_n(z)$ in (3.4.2)) are called *OPE coefficients*. OPE coefficients of Fock space fields are Fock space fields (as functions of z). In particular, if X is *holomorphic*, then we define the $*_n$ product by $X *_n Y = C_n$. We write $*$ for $*_0$ and call $X * Y$ the *OPE multiplication*, or the *OPE product* of X and Y. In fact, we have the following formula.

Proposition 3.4.1. *We have*

$$\Phi^{\odot n} = (2c)^{n/2} H_n^*\left(\frac{\Phi}{\sqrt{2c}}\right), \tag{3.4.3}$$

where $H_n(z) = \sum_{k=0}^n a_k z^k$ are the n'th Hermite polynomials with generating function,

$$e^{tx - \frac{1}{2}t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x).$$

Proof. By Wick's formula, $\Phi(\zeta)\Phi^{\odot n}(z) = 2nG(\zeta, z)\Phi^{\odot(n-1)}(z) + \Phi^{\odot(n+1)}(z) +$

$o(1)$, we have

$$\Phi * \Phi^{\odot n} = 2cn\Phi^{\odot(n-1)} + \Phi^{\odot(n+1)}.$$

Using induction argument and recurrence relation

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x),$$

we prove (3.4.3) for $\Phi^{\odot(n+1)}$. □

By definition,

$$e^{*\alpha\Phi} = \sum_{n=0}^{\infty} \alpha^n \frac{\Phi^{*n}}{n!}.$$

Proposition 3.4.2. *We have*

$$e^{*\alpha\Phi} = C^{\alpha^2} e^{\odot\alpha\Phi},$$

where C is conformal radius.

Proof. Using the generating function for the Hermite polynomials, we have

$$e^{\odot\alpha\Phi} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\Phi^{\odot n} = \sum_{n=0}^{\infty} \frac{\alpha^n (2c)^{\frac{1}{2}n}}{n!} H_n^*\left(\frac{\Phi}{\sqrt{2c}}\right) = e^{*\alpha\Phi} e^{-c\alpha^2}.$$

□

3.5 Conformal Fock space fields

This section is borrowed from [12, Lectures 4.2 – 4.4]. We use Lie derivative of a conformal field to define the stress tensor and to state Ward's identities.

A general conformal Fock space field is a linear combination of basic fields X_α,

$$X = \sum_\alpha f_\alpha X_\alpha,$$

where f_α's are non-random conformal fields, see Section 2.1. A non-random conformal field f is said to be *invariant* with respect to some conformal automorphism τ of M if

$$(f \,\|\, \phi) = (f \,\|\, \phi \circ \tau^{-1})$$

for all charts ϕ. For example, suppose D is a planar domain and let us write f for $(f \,\|\, \mathrm{id}_D)$. Then f is a τ-invariant $[\lambda, \lambda_*]$-differential if

$$f(z) = f(\tau z)\ \tau'(z)^\lambda\ \overline{\tau'(z)}^{\lambda_*}.$$

It is because τ is the transition map between the charts $\phi \circ \tau^{-1}$ and $\phi = \mathrm{id}_D$. By definition, a *random* conformal field (or a family of conformal fields) is *τ-invariant* if all correlations are invariant as non-random conformal fields.

Suppose a non-random smooth vector field v is holomorphic in some open set $U \subset M$. For a conformal Fock space field X, we define the Lie derivative $\mathcal{L}_v X$ in U as

$$(\mathcal{L}_v X \,\|\, \phi) = \frac{d}{dt}\Big|_{t=0} (X \,\|\, \phi \circ \psi_{-t}),$$

where ψ_t is a local flow of v, and ϕ is an arbitrary chart.

Proposition 3.5.1. *If X is a $[\lambda, \lambda_*]$-differential, then*

$$\mathcal{L}_v X = \left(v\partial + \bar v \bar\partial + \lambda v' + \lambda_* \overline{v'}\right) X;$$

if X is a pre-Schwarzian form of order μ

$$\mathcal{L}_v X = \left(v\partial + v'\right) X + \mu v'';$$

if X is a Schwarzian form of order μ

$$\mathcal{L}_v X = \left(v\partial + 2v'\right) X + \mu v'''.$$

Proof. We denote the fields X_t by

$$(X_t \| \phi)(z) = (X \| \phi \circ \psi_{-t})(z) \quad z \in \phi U, \quad |t| \ll 1,$$

where ϕ is the given chart. For example, if X is a $[\lambda, \lambda_*]$-differential, then

$$(X_t \| \phi)(z) = (X(\psi_t z) \| \psi_{-t}) = (\psi_t'(z))^\lambda (\overline{\psi_t'(z)})^{\lambda_*} (X \| \phi)(\psi_t z),$$

and if X is a Schwarzian form of order μ, then

$$(X_t\|\phi)(z) = (\psi_t'(z))^2(X\|\phi)(\psi_t z) + \mu S_{\psi_t}(z).$$

Since ψ is a local flow of v, we have $\psi_0' = 1, \dot{\psi}_0 = v, \dot{\psi}'_0 = v', (\dot{N}_\psi)_0 = v''$ and $(\dot{S}_\psi)_0 = v'''$. Differentiate above equations, we get the results. $\square$

We recall basic properties of Lie derivatives:

- $\mathcal{L}_v$ is an $\mathbb{R}$-linear operator on Fock space fields;
- $\mathbf{E}[\mathcal{L}_v X] = \mathcal{L}_v(\mathbf{E}[X])$;
- $\mathcal{L}_v(\bar{X}) = \overline{(\mathcal{L}_v X)}$;
- $\mathcal{L}_v(\partial X) = \partial(\mathcal{L}_v X)$ and $\mathcal{L}_v(\bar{\partial} X) = \bar{\partial}(\mathcal{L}_v X)$;
- Leibniz's rule applies to Wick's products, OPE products, and tensor products.

We define the $\mathbb{C}$-linear part $\mathcal{L}_v^+$ and anti-linear part $\mathcal{L}_v^-$ of the Lie derivative $\mathcal{L}_v$ by

$$2\mathcal{L}_v^+ = \mathcal{L}_v - i\mathcal{L}_{iv}, \qquad 2\mathcal{L}_v^- = \mathcal{L}_v + i\mathcal{L}_{iv}.$$

3.6 Stress tensor

This section is borrowed from [12, Lectures 5.2 – 5.3]. A Fock space field X in D is said to have a (symmetric) stress tensor $(A, \bar{A})$ ($X \in \mathcal{F}(A, \bar{A})$) if A is a holomorphic quadratic differential and if Ward's OPE holds for X, i.e., on a given chart $\phi : U \to \phi U$,

$$\mathrm{Sing}_{\zeta\to z}[A(\zeta)X(z)] = (\mathcal{L}_{k_\zeta}^+ X)(z), \quad \mathrm{Sing}_{\zeta\to z}[A(\zeta)\bar{X}(z)] = (\mathcal{L}_{k_\zeta}^+ \bar{X})(z),$$

where the (local) vector field k_ζ is defined by $(k_\zeta \parallel \phi)(\eta) = 1/(\zeta - \eta)$. Ward's family $\mathcal{F}(A, \bar{A})$ is closed under differentiation and OPE multiplication, see [12, Proposition 5.8]. In the case of differentials or forms, it is enough to verify Ward's OPEs in just one chart. For example, a $[\lambda, \lambda_*]$-differential X

is in $\mathcal{F}(A, \bar{A})$ if and only if the following operator product expansions hold in every/some chart:

$$A(\zeta)X(z) \sim \frac{\lambda X(z)}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z}, \quad A(\zeta)\bar{X}(z) \sim \frac{\bar{\lambda}_* \bar{X}(z)}{(\zeta - z)^2} + \frac{\partial \bar{X}(z)}{\zeta - z}.$$

Let X be a form of order μ. Then $X \in \mathcal{F}(A, \bar{A})$ if and only if the following operator product expansion holds in every/some chart:

$$A(\zeta)X(z) \sim \frac{\mu}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z} \qquad \text{for a pre-pre-Schwarzian form } X;$$
$$A(\zeta)X(z) \sim \frac{2\mu}{(\zeta - z)^3} + \frac{X(z)}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z} \qquad \text{for a pre-Schwarzian form } X;$$
$$A(\zeta)X(z) \sim \frac{6\mu}{(\zeta - z)^4} + \frac{2X(z)}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z} \qquad \text{for a Schwarzian form } X.$$

For example, Gaussian free field $\Phi_{(0)}$ has a stress tensor

$$A_{(0)} = -\frac{1}{2} J_{(0)} \odot J_{(0)}, \qquad J_{(0)} = \partial \Phi_{(0)}.$$

Precisely, in the identity chart of upper half-plane, we have

$$\begin{aligned} A_{(0)}(\zeta)\Phi_{(0)}(z) &= -\mathbf{E}[J_{(0)}(\zeta)\Phi_{(0)}(z)]J_{(0)}(\zeta) - \frac{1}{2} J_{(0)}(\zeta) \odot J_{(0)}(\zeta) \odot \Phi_{(0)}(z) \\ &\sim \frac{J(z)}{\zeta - z} \quad \text{as } \zeta \to z. \end{aligned}$$

Therefore we conclude:

Proposition 3.6.1. *We have $\Phi \in \mathcal{F}(A, \overline{A})$.*

Moreover, the Leibniz's rule for Lie derivative implies the following proposition.

Proposition 3.6.2. *All OPE coefficients of fields in $\mathcal{F}(A, \overline{A})$ belong to $\mathcal{F}(A, \overline{A})$. In particular, if $X \in \mathcal{F}(A, \overline{A})$, then $\partial X \in \mathcal{F}(A, \overline{A})$.*

This holomorphic quadratic differential A coincides with the Virasoro field T in the upper half-plane uniformization. While A itself does not belong to $\mathcal{F}(A, \bar{A})$, the Virasoro field T is in $\mathcal{F}(A, \bar{A})$. We review the abstract theory of Virasoro field in the next section.

3.7 Virasoro field

This section is borrowed from [12, Lecture 7 and Appendix 11]. A Fock space field T is said to be the *Virasoro field* for Ward's family $\mathcal{F}(A, \bar{A})$ if

- $T \in \mathcal{F}(A, \bar{A})$, and
- $T - A$ is a non-random holomorphic Schwarzian form.

For example, $T_{(0)} := -\frac{1}{2} J_{(0)} * J_{(0)}$ is the Virasoro field for the Gaussian free field with central charge $c = 1$. To express $T_{(0)}$ in terms of stress tensor and non-random field, we need the *Schwarzian* of D:

$$S(z) = S(z, z), \quad S(\zeta, z) := -12 \partial_\zeta \partial_z u(\zeta, z),$$

where $u(\zeta, z) = G(\zeta, z) + \log|\zeta - z|$, as usual. Then,we have

$$\mathbf{E}[J_{(0)}(\zeta) J_{(0)}(z)] \ = 2\partial_\zeta \partial_z G(\zeta, z) = -\frac{1}{(\zeta - z)^2} - \frac{1}{6} S(\zeta, z),$$

therefore,

$$T_{(0)} = -\frac{1}{2} J_{(0)} \odot J_{(0)} + \frac{1}{12} S.$$

We define Virasoro primary fields and current primary fields in terms of Virasoro generators L_n and current generators J_n:

$$L_n(z) := \frac{1}{2\pi i} \oint_{(z)} (\zeta - z)^{n+1} T(\zeta) d\zeta, \tag{3.7.1}$$

$$J_n(z) := \frac{1}{2\pi i} \oint_{(z)} (\zeta - z)^{n} J(\zeta) d\zeta. \tag{3.7.2}$$

Proposition 3.7.1 (Proposition 7.5 in [12]). *Let X be a Fock space field. Any two of the following assertions imply the third one (but neither one implies the other two):*

- $X \in \mathcal{F}(A, \bar{A})$*;*
- X *is a* $[\lambda, \lambda_*]$*-differential;*
- $L_{\ge 1} X = 0,\ L_0 X = \lambda X,\ L_{-1} X = \partial X,$ *and similar equations hold for* $\bar{X}$.

Here, $L_{\geq k}X = 0$ means that $L_nX = 0$ for all $n \geq k$. We call fields satisfying all three conditions *(Virasoro) primary* fields in $\mathcal{F}(A, \bar{A})$.

A (Virasoro) primary field X is called *current primary* if

$$J_{\geq 1}X = J_{\geq 1}\bar{X} = 0,$$

and

$$J_0X = -iqX, \quad J_0\bar{X} = i\bar{q}_*\bar{X}$$

for some numbers q and q_*. They are called "charges" of X. Here, $J_{\geq k}X = 0$ means that $J_nX = 0$ for all $n \geq k$. We use the following proposition to prove Proposition 5.6.1 (level two degeneracy equations for Ψ).

Proposition 3.7.2 (Proposition 11.2 in [12])**.** *For a current primary field V with charges q, q_* in $\mathcal{F}_{(b)}$,*

$$\Big(L_{-2} - \frac{1}{2q^2}L_{-1}^2\Big)V = 0$$

provided $2q(b+q) = 1$.

Chapter 4

Dipolar CFT

After we discuss central charge modifications of the Gaussian free field in a simply connected domain D with two marked boundary points $q_\pm$, we define a dipolar version of the Ward functional in terms of the Lie derivative. Based on this approach, we derive Ward equations in the dipolar case.

4.1 Central charge modification

For a given simply connected domain D with two marked points $q_\pm \in \partial D$, we consider a conformal transformation

$$w : (D, q_-, q_+) \to (\mathbb{H}, -1, 1)$$

from D onto the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\}$. For a parameter $b = \sqrt{\kappa/8} - \sqrt{2/\kappa}$, we define the central charge modifications $\Phi \equiv \Phi_{(b)}$ of the Gaussian free field $\Phi_{(0)}$ with the Dirichlet boundary condition by

$$\Phi_{(b)} = \Phi_{(0)} + \varphi, \qquad \varphi = -2b \arg(\frac{w'}{1-w^2}).$$

It is well-defined since φ does not depend on the choice of w. We also define the current field $J \equiv J_{(b)}$ by

$$J_{(b)} = \partial \Phi_{(b)} = J_{(0)} + j, \qquad j = ib(\frac{w''}{w'} + \frac{2ww'}{1-w^2}).$$

The current field is an $\mathrm{Aut}(D, q_-, q_+)$-invariant pre-Schwarzian form of order ib.

Remark. In the chordal theory [12, Lecture 11], the central charge modification was defined as

$$\Phi_{(b)}^{\text{chordal}} = \Phi_{(0)} - 2b \arg w',$$

where w is a conformal transformation from (D, q) onto $(\mathbb{H}, \infty)$ for a marked boundary point $q \in \partial D$. Also, the radial version of central charge modification was defined as

$$\Phi_{(b)}^{\text{radial}} = \Phi_{(0)} - 2b \arg \frac{w'}{w},$$

where w is a conformal transformation from (D, q) onto $(\mathbb{D}, 0)$ for a marked interior point $q \in D$, see [11].

Let $\mathcal{F}_{(b)}$ denote the OPE family of the field $\Phi_{(b)}$, the algebra spanned by the generators $1, \partial^j \bar{\partial}^k \Phi_{(b)}$ and $\partial^j \bar{\partial}^k e^{*\alpha\Phi_{(b)}}$ under OPE multiplication. Since the OPE product of two conformally invariant fields is conformally invariant, the fields is $\mathcal{F}_{(b)}$ are invariant with respect to $\mathrm{Aut}(D, q_-, q_+)$. Compare the following proposition to [12, Proposition 10.1]. See Sections 3.6 – 3.7 for the definitions of a stress tensor and the Virasoro field.

Proposition 4.1.1. *The bosonic field $\Phi_{(b)}$ has a stress tensor, and the Virasoro field is given by*

$$T \equiv T_{(b)} = -\frac{1}{2} J * J + ib\partial J. \tag{4.1.1}$$

Proof. Let us define a holomorphic field A by

$$A \equiv A_{(b)} = A_{(0)} + (ib\partial - j)J_{(0)}, \qquad A_{(0)} = -\frac{1}{2} J_{(0)} \odot J_{(0)}. \tag{4.1.2}$$

Then A is a quadratic differential. Indeed, as in the chordal and the radial cases, $ib\partial J_{(0)}$ and $jJ_{(0)}$ satisfy the following transformation laws:

$$\begin{aligned} ib\partial J_{(0)} &= ibh'' \tilde{J}_{(0)} \circ h + ib(h')^2 \partial \tilde{J}_{(0)} \circ h, \\ jJ_{(0)} &= ib\Big(\frac{h''}{h'}\Big) h' \tilde{J}_{(0)} \circ h + (h')^2 (\tilde{j}\tilde{J}_{(0)}) \circ h, \end{aligned}$$

where $J_{(0)} \equiv (J_{(0)} \| \phi), \tilde{J}_{(0)} \equiv (J_{(0)} \| \tilde{\phi})$ and h is the transition map $\tilde{\phi} \circ \phi^{-1}$. Since $\Phi_{(b)}$ is a real part of pre-pre-Schwarzian form of order ib, it is enough to check Ward's OPE in the $(\mathbb{H},-1,1)$-uniformization,

$$A(\zeta)\Phi(z)\sim \frac{ib}{(\zeta-z)^2}+\frac{J_{(b)}}{\zeta-z}.$$

However, this is immediate from Ward's OPE in the case $b=0$ and the following operator product expansions:

$$\partial J_{(0)}(\zeta)\Phi_{(0)}(z)\sim \frac{1}{(\zeta-z)^2},\qquad j(\zeta)J_{(0)}(\zeta)\Phi_{(0)}(z)\sim -\frac{j(z)}{\zeta-z}.$$

Finally, let us show that $T_{(b)}$ is the Virasoro field for the OPE family $\mathcal{F}_{(b)}$. Since $\mathcal{F}_{(b)}$ is closed under differentiation and OPE multiplication (see Section 3.6 or [12, Proposition 5.8]), $T_{(b)}\in\mathcal{F}_{(b)}$. Therefore, it has a stress tensor $(A_{(b)},\overline{A_{(b)}})$. It follows from the expressions of T and J that

$$T=A+\frac{1}{12}S_w-\frac{j^2}{2}+ibj',$$

where $S_w=(w''/w)'-\frac{1}{2}(w''/w')^2$ is the Schwarzian derivative of w. The term $-\frac{1}{2}j^2+ibj'$ simplifies $-b^2S_w-2b^2w'^2/(1-w^2)^2$. Thus we get

$$T=A+\frac{1-12b^2}{12}S_w-2b^2(\frac{w'}{1-w^2})^2 \tag{4.1.3}$$

and therefore T is a Schwarzian form of order $\frac{1}{12}c$. The central charge c is given by

$$c\equiv 1-12b^2=\frac{(3\kappa-8)(6-\kappa)}{2\kappa}.$$

□

4.2 Ward's functionals and Ward's identities

In this section we modify the definition of Ward's functionals in the chordal case (see [12, Sections 5.5 – 5.6]) into the dipolar case and derive Ward's identities. For a given open set U such that $\bar U\subset \bar D\setminus\{q_\pm\}$ and a smooth vector field v on $\bar U$, Ward's functional $W^\pm(v;U)$ is defined by

$$W^+(v;U)=\frac{1}{2\pi i}\int_{\partial U}vA-\frac{1}{\pi}\iint_U(\bar\partial v)A,\qquad W^-(v;U)=\overline{W^+(v;U)}$$

where $A \equiv A_{(b)} = A_{(0)} + (ib\partial - j)J_{(0)}$, $A_{(0)} = -\frac{1}{2}J_{(0)} \odot J_{(0)}$, see (4.1.2), and $j = \mathbf{E}[J_{(b)}]$. We also write $W(v;U) = 2\,\mathrm{Re}\,W^+(v;U)$. Then $\mathbf{E}\,[W^+(v;U)\,\mathcal{X}]$ is a well-defined correlation function if $\mathcal{X}$ is a Fock space functional on $\bar{D} \setminus \{q_\pm\}$ with nodes in U and in the maximal open set $D_{\mathrm{hol}}(v)$ where v is holomorphic.

Recall that the following statements are equivalent (see [12, Propositions 5.3 and 5.10]):

- Ward's OPE holds for a Fock space field X.
- The residue form of Ward's identity for X

  $$\mathcal{L}_v X(z) = \frac{1}{2\pi i}\oint_{(z)} vA^+\,X(z) - \frac{1}{2\pi i}\oint_{(z)} \bar{v}A^-\,X(z)$$

  holds on $D_{\mathrm{hol}}(v) \cap U$ for all (local) smooth vector field v. (See Section 3.5 for the definition of Lie derivatives $\mathcal{L}_v$ and their basic properties.)
- For all $z \in D_{\mathrm{hol}}(v) \cap U$

  $$\mathbf{E}[\mathcal{Y}\mathcal{L}(v,U)X(z)] = \mathbf{E}[W(v,U)X(z)\mathcal{Y}]$$

  holds for all correlation functional $\mathcal{Y}$ whose nodes are in $(D \setminus \bar{U})$.

The definition of Ward's functionals can be extended to meromorphic vector fields. For a meromorphic vector field v which is continuous up to the boundary and has a simple zero at $q_\pm$, we define Ward's functional $W^+(v;\bar{D} \setminus \{q_\pm\})$ by

$$W^+(v;\bar{D} \setminus \{q_\pm\}) = \lim_{\varepsilon \to 0} W^+(v;D_\varepsilon),$$

where $D_\varepsilon = D \setminus (B(q_-,\varepsilon) \cup B(q_+,\varepsilon) \cup \bigcup_j B(p_j,\varepsilon))$ and p_j's are poles of v. Thus we have

$$W^+(v;\bar{D} \setminus \{q_\pm\}) = \frac{1}{2\pi i}\int_{\partial D} vA - \frac{1}{\pi}\iint_D (\bar{\partial} v)A,$$

where $\bar{\partial} v$ is considered as a distribution. Note that vA has a removable singularity at $q_\pm$. Suppose X_j's are in the OPE family $\mathcal{F}_{(b)}$. Then the global

Ward's identity

$$\mathbf{E}\left[\mathcal{L}_v X_1(z_1)\cdots X_n(z_n)\right] = \mathbf{E}\left[W(v;\bar{D}\setminus\{q_\pm\})X_1(z_1)\cdots X_n(z_n)\right] \qquad (4.2.1)$$

holds if all $z_j \in D_{\mathrm{hol}}(v)$. Compare (4.2.1) to [12, Proposition 5.9].

We now represent a quadratic differential $A \equiv A_{(b)} = A_{(0)} + (ib\partial - j)J_{(0)}$ in terms of Ward's functionals associated with the dipolar Loewner vector field v_ζ :

$$(v_\zeta \,\|\, \mathrm{id}_{\overline{\mathbb{H}}\setminus\{\pm 1\}})(z) = \frac{1-z^2}{2}\frac{1-\zeta z}{\zeta - z}.$$

Proposition 4.2.1. *In the identity chart* $\mathrm{id}_{\mathbb{H}}$ *of* $\mathbb{H}$, *we have*

$$\mathbf{E}[A(\zeta)\mathcal{X}] = \frac{2}{(1-\zeta^2)^2}\mathbf{E}[\Big(W^+(v_\zeta;\overline{\mathbb{H}}\setminus\{\pm 1\}) + W^-(v_{\bar\zeta};\overline{\mathbb{H}}\setminus\{\pm 1\})\Big)\mathcal{X}]$$

for all $\mathcal{X} \in \mathcal{F}_{(b)}$.

Proof. For $\zeta \in \mathbb{H}$, by the definition of Ward's functional,

$$W^+(v_\zeta;\overline{\mathbb{H}}\setminus\{\pm 1\}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} v_\zeta A - \frac{1}{\pi}\int_{\mathbb{H}}(\bar\partial v_\zeta)A.$$

On the other hand, the reflected vector field

$$v_\zeta^{\#}(z) := \overline{v_\zeta(\bar z)} = v_{\bar\zeta}(z)$$

is holomorphic in $\mathbb{H}$. Therefore, we have

$$W^+(v_{\bar\zeta};\overline{\mathbb{H}}\setminus\{\pm 1\}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} v_{\bar\zeta}A.$$

Since A is real on the boundary and $\overline{v_{\bar\zeta}} = v_\zeta$ on the boundary,

$$W^-(v_{\bar\zeta};\overline{\mathbb{H}}\setminus\{\pm 1\}) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} v_\zeta A.$$

Proposition now follows from the fact that $\bar\partial v_\zeta = -\frac{1}{2}\pi(1-\zeta^2)^2\delta_\zeta$. Note that above equations hold in correlation with $\mathcal{X} \in \mathcal{F}_{(b)}$. □

4.3 Ward's equations in the upper half-plane

We will use Ward's equations below (Proposition 4.3.1) to prove Theorem 2.2.1. For $Y \in \mathcal{F}_{(b)}$, let us express its (holomorphic part of) Lie derivative $\mathcal{L}^+_{v_\zeta} Y$ in terms of the singular part of operator product expansion $A(\zeta)Y(z) \sim \sum_{j \le -1} C_j(z)(\zeta - z)^j$ as $\zeta \to z$ and the OPE coefficients C_j $(j = -1, -2, -3)$:

$$\mathcal{L}^+_{v_\zeta} Y(z) = \sum_{j=-3}^{-1} P_j(\zeta, z)\, C_j(z) + \frac{(1-\zeta^2)^2}{2} \sum_{j \le -1} C_j(z)(\zeta - z)^j, \qquad (4.3.1)$$

where $P_{-1}(\zeta, z) = \frac{1}{2}(2\zeta + z - \zeta^3 - \zeta^2 z - \zeta z^2)$, $P_{-2}(\zeta, z) = \frac{1}{2}(1 - \zeta^2 - 2\zeta z)$, $P_{-3}(\zeta, z) = -\frac{1}{2}\zeta$. Equation (4.3.1) can be shown by the identities

$$\mathcal{L}^+_{v_\zeta} Y(z) = \frac{1}{2\pi i} \oint_{(z)} v_\zeta(\eta) A(\eta) Y(z)\, d\eta = \sum_{j \le -1} C_j(z) \frac{1}{2\pi i} \oint_{(z)} (\eta - z)^j\, v_\zeta(\eta)\, d\eta,$$

and

$$\frac{1}{2\pi i} \oint_{(z)} (\eta - z)^j\, v_\zeta(\eta)\, d\eta = \frac{(1-\zeta^2)^2}{2} (\zeta - z)^j + P_j(\zeta, z) \qquad (j \le -1)$$

if we set $P_j(\zeta, z) \equiv 0$ for $j \le -4$.

We state Ward's equation in terms of Virasoro generators L_n. Let us recall the definition of L_n :

$$L_n(z) := \frac{1}{2\pi i} \oint_{(z)} (\zeta - z)^{n+1} T(\zeta)\, d\zeta. \qquad (4.3.2)$$

As operators acting on fields, the modes L_n can be viewed as OPE multiplications, $L_n X = T *_{-n-2} X$. See Section 3.4 for the definition of $*_n$.

Proposition 4.3.1. *For $Y \in \mathcal{F}_{(b)}$ and $X = X_1(z_1) \cdots X_n(z_n)$ $(X_j \in \mathcal{F}_{(b)})$,*

$$\begin{aligned} &\mathbf{E}\,[Y(z)\mathcal{L}^+_{v_z} X] + \mathbf{E}\,[\mathcal{L}^-_{v_{\bar z}}(Y(z)X)] \\ &= \frac{(1-z^2)^2}{2} \mathbf{E}\,[(L_{-2}Y)(z)X] - \frac{3z(1-z^2)}{2} \mathbf{E}\,[(L_{-1}Y)(z)X] \\ &+ \frac{3z^2-1}{2} \mathbf{E}\,[(L_0 Y)(z)X] + \frac{z}{2} \mathbf{E}\,[(L_1 Y)(z)X] + b^2 \mathbf{E}\,[Y(z)X], \end{aligned}$$

where all fields are evaluated in the identity chart of $\mathbb{H}$.

Proof. Since $Y \in \mathcal{F}_{(b)}$, we use (4.3.1) and subtract the singular part of OPE. As the result we get

$$\begin{aligned}\mathbf{E}[(A * Y)(z)X] = \lim_{\zeta \to z} \mathbf{E}[A(\zeta)Y(z)X] - \frac{2}{(1-\zeta^2)^2}\mathbf{E}\left[(\mathcal{L}^+_{v_\zeta} Y(z))X\right] \\ + \frac{2}{(1-z^2)^2}\Big(\frac{3z(1-z^2)}{2}\mathbf{E}\left[(A *_{-1} Y)(z)X\right] \\ + \frac{1-3z^2}{2}\mathbf{E}\left[(A *_{-2} Y)(z)X\right] - \frac{z}{2}\mathbf{E}\left[(A *_{-3} Y)(z)X\right]\Big).\end{aligned}$$

We now use Proposition 4.2.1 and Leibniz's rule for Lie derivatives to derive

$$\begin{aligned}&\mathbf{E}\left[Y(z)\mathcal{L}^+_{v_z} X\right] + \mathbf{E}\left[\mathcal{L}^-_{v_{\bar z}}(Y(z)X)\right] \\ &\quad = \frac{(1-z^2)^2}{2}\mathbf{E}\left[(A * Y)(z)X\right] - \frac{3z(1-z^2)}{2}\mathbf{E}\left[(A *_{-1} Y)(z)X\right] \\ &\quad + \frac{3z^2-1}{2}\mathbf{E}\left[(A *_{-2} Y)(z)X\right] + \frac{z}{2}\mathbf{E}\left[(A *_{-3} Y)(z)X\right].\end{aligned}$$

Proposition now follows since $T = A - 2b^2/(1-z^2)^2$ in the identity chart of $\mathbb{H}$. $\square$

Chapter 5

Vertex fields

In the next chapter the boundary condition changing operator Ψ will be introduced as a vertex field rooted at two marked boundary points $q_\pm$. Also we expand our collection of the OPE family of $\Phi_{(b)}$ by considering the (rooted) multi-vertex fields with the neutrality condition. For this purpose, we first introduce the formal bosonic fields and then we define the multi-vertex fields in terms of the formal bosonic fields.

5.1 Formal fields

The formal 1-point fields $\Phi^+_{(0)}$ and $\Phi^-_{(0)}$ can be interpreted as the "holomorphic part" and the "anti-holomorphic part" of the Gaussian free field $\Phi_{(0)}$ in the sense that $\Phi_{(0)} = \Phi^+_{(0)} + \Phi^-_{(0)}$ and $\Phi^-_{(0)} = \overline{\Phi^+_{(0)}}$. By definition they have the following formal correlations

$$\mathbf{E}[\Phi^+_{(0)}(z)\Phi^+_{(0)}(z_0)] = \log\frac{1}{w(z)-w(z_0)},$$
$$\mathbf{E}[\Phi^+_{(0)}(z)\Phi^-_{(0)}(z_0)] = \log(w(z)-\overline{w(z_0)}),$$

where w is any conformal transformation from D onto $\mathbb{H}$. Of course, neither $\Phi^+_{(0)}$ nor $\Phi^-_{(0)}$ is a genuine Fock space field. However, the formal field

$$\Phi_{(0)}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*] := \sum_{j=1}^{n}\sigma_j\Phi^+_{(0)}(z_j) - \sigma_{*j}\Phi^-_{(0)}(z_j) \tag{5.1.1}$$

is a well-defined (multivalued) Fock space field if and only if the "neutrality condition"

$$\int \boldsymbol{\sigma} + \boldsymbol{\sigma}_* := \sum_j (\sigma_j + \sigma_{*j}) = 0 \tag{5.1.2}$$

holds. Let (NC_0) denote the condition (5.1.2). Here $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_*$ are divisors. More precisely, $\boldsymbol{\sigma}$ (and also $\boldsymbol{\sigma}_*$) is a function $\boldsymbol{\sigma} : D \cup \partial D \mapsto \mathbb{C}$ which takes the value 0 at all points except for finitely many points. Hence we can denote

$$\boldsymbol{\sigma} = \sum_{j=1}^{n} \sigma_j \cdot z_j \quad \text{and} \quad \boldsymbol{\sigma}_* = \sum_{j=1}^{n} \sigma_{*j} \cdot z_j,$$

where $z_j \in D \cup \partial D$ and $\sigma_j = \sigma_{z_j} = \boldsymbol{\sigma}(z_j), \sigma_{*j} = \sigma_{*z_j} = \boldsymbol{\sigma}_*(z_j)$. Sometimes it is convenient to consider $\boldsymbol{\sigma}$ as an atomic measure;

$$\int \boldsymbol{\sigma} = \int \sum_j \sigma_j d\delta_{z_j} = \sum \sigma_j.$$

For example, as a bi-variant field, $\Phi^+_{(0)}(z, z_0) = \Phi^+_{(0)}(z) - \Phi^+_{(0)}(z_0)$ is a multivalued Fock space field:

$$\Phi^+_{(0)}(z, z_0) = \{\Phi^+_{(0)}(\gamma) := \int_\gamma J_{(0)}(\zeta)\, d\zeta \,|\, \gamma \text{ is a curve from } z_0 \text{ to } z\}.$$

Lemma 5.1.1. *If a double divisor $(\boldsymbol{\sigma}, \boldsymbol{\sigma_*})$ satisfies the neutrality condition (NC_0), then the formal bosonic field $\Phi[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}]$ can be represented as a linear combination of well-defined Fock space fields.*

Proof. Let us choose any point $z_0 \in D$. Then

$$\Phi[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}] = \Phi^+(z_0) \int \boldsymbol{\sigma} - \Phi^-(z_0) \int \boldsymbol{\sigma_*} + \sum \sigma_j \Phi^+(z_j, z_0) - \sigma_{*j} \Phi^-(z_j, z_0).$$

Under the neutrality condition, the first two terms in the right-hand side become the Fock space correlation functional:

$$\Phi^+(z_0) \int \boldsymbol{\sigma} - \Phi^-(z_0) \int \boldsymbol{\sigma_*} = \Phi(z_0) \int \boldsymbol{\sigma}.$$

□

Also, we define $\Phi^{\pm}_{(b)}$ by

$$\Phi^+_{(b)}(z) = \Phi^+_{(0)}(z) + ib\log\frac{w'_z}{1-w_z^2}, \quad \Phi^-_{(b)}(z) = \Phi^-_{(0)}(z) - ib\log\frac{\bar w'_z}{1-\bar w_z^2}$$

for all $z \in \overline{D} \setminus \{q_\pm\}$, and

$$\Phi^+_{(b)}(q_\pm) := \Phi^+ * 1(q_\pm) = \Phi^+_{(0)}(q_\pm),$$
$$\Phi^-_{(b)}(q_\pm) := \Phi^- * 1(q_\pm) = \Phi^-_{(0)}(q_\pm),$$

where 1 is the constant field.

5.2 Multi-vertex fields

In this section, we construct multi-vertex fields

$$\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \equiv e^{*i\Phi_{(b)}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]}$$

and express their correlation functions. Given a double divisor $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*)$ with the neutrality condition (NC_0), we define

$$c[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] := \frac{1}{2}\mathbf{E}[\Phi_{(0)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]^{*2}].$$

Let w be a conformal map from D onto $\mathbb{H}$ and denote $w_j = w(z_j), w'_j = w'(z_j)$. It is easy to check that

$$\begin{aligned} c[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] = &-\frac{1}{2}\sum_{j=1}^n \sigma_j^2 \log w'_j + \sigma_{*j}^2 \log \bar w'_j \\ &- \sum_{j<k}\Big(\sigma_j\sigma_k\log(w_j - w_k) + \sigma_{*j}\sigma_k\log(\bar w_j - w_k) \\ &\qquad + \sigma_j\sigma_{*k}\log(w_j - \bar w_k) + \sigma_{*j}\sigma_{*k}\log(\bar w_j - \bar w_k)\Big). \end{aligned}$$

Applying Proposition 3.4.2, we have the following formula.

Proposition 5.2.1. *If a double divisor $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*)$ satisfies the neutrality condition (NC_0), then*

$$\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \equiv e^{*i\Phi_{(b)}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]} = e^{-c[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]}e^{i\mathbf{E}\Phi_{(b)}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]}\mathcal{V}^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*],$$

where $\mathcal{V}^{\odot}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*] = e^{\odot i\Phi_{(0)}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]}$. *In particular,* $\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]$ *belongs to* $\mathcal{F}_{(b)}$ *and is a* (h_j, h_{*j})*-differential with respect to* z_j. *Here*

$$(h_j, h_{*j}) = (\frac{\sigma_j^2}{2} - \sigma_j b, \frac{\sigma_{*j}^2}{2} - \sigma_{*j} b).$$

More precisely, in the $(\mathbb{H},-1,1)$*-uniformization with conformal transformation* $w : (D, q_-, q_+) \mapsto (\mathbb{H},-1,1)$, *we have*

$$\mathbf{E}\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*] = \prod_{j=1}^{n} M^{(\sigma_j,\sigma_{*j})}(z_j) \prod_{j<k} I_{j,k}(z_j, z_k),$$

where

$$M^{(\sigma,\sigma_*)}(z) = (w'(z))^{\sigma^2/2-\sigma b}(\overline{w}'(z))^{\sigma_*^2/2-\sigma_* b}(1-w(z)^2)^{\sigma b}(1-\overline{w}(z)^2)^{\sigma_* b}, \tag{5.2.1}$$

and

$$\begin{aligned} I_{j,k} = {} & (w(z_j) - w(z_k))^{\sigma_j\sigma_k}(w(z_j) - \overline{w}(z_k))^{\sigma_j\sigma_{*k}} \\ & \times (\overline{w}(z_j) - w(z_k))^{\sigma_{*j}\sigma_k}(\overline{w}(z_j) - \overline{w}(z_k))^{\sigma_{*j}\sigma_{*k}}. \end{aligned} \tag{5.2.2}$$

For example, a chiral bi-vertex field is defined by

$$\begin{aligned} &\mathcal{O}^{(\sigma)}(z, z_0) \equiv \mathcal{O}[\sigma \cdot z - \sigma \cdot z_0] = e^{*i\sigma\Phi^+_{(b)}(z,z_0)} \\ &= (w'(z))^{\frac{\sigma^2}{2}-\sigma b}(w'(z_0))^{\frac{\sigma^2}{2}+\sigma b}\left(\frac{1-w(z)^2}{1-w(z_0)^2}\right)^{\sigma b}(w(z)-w(z_0))^{-\sigma^2} e^{\odot i\sigma\Phi^+_{(0)}(z,z_0)}, \end{aligned} \tag{5.2.3}$$

where w is a conformal map $w : (D, q_-, q_+) \mapsto (\mathbb{H},-1,1)$.

The following proposition induces that multi-vertex fields belong to the Ward's family $\mathcal{F}(A,\overline{A})$, hence, $\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]$ is a primary field in $\mathcal{F}(A,\overline{A})$.

Proposition 5.2.2. *Under the neutrality condition,* $\mathcal{O} \equiv \mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]$ *are well-defined* $\mathrm{Aut}(D, q_\pm)$*-invariant Fock space fields. Moreover, they satisfy Ward's OPE:*

$$T(\zeta)\mathcal{O}(\boldsymbol{z}) \sim h_j\,\frac{\mathcal{O}(\boldsymbol{z})}{(\zeta - z_j)^2} + \frac{\partial_{z_j}\mathcal{O}(\boldsymbol{z})}{\zeta - z_j}, \qquad T(\zeta)\bar{\mathcal{O}}(\boldsymbol{z}) \sim \bar{h}_{*j}\frac{\bar{\mathcal{O}}(\boldsymbol{z})}{(\zeta - z_j)^2} + \frac{\partial_{z_j}\bar{\mathcal{O}}(\boldsymbol{z})}{\zeta - z_j},$$

as $\zeta \to z_j$.

Proof. Under the neutrality condition, $\sum \sigma_j \Phi^+_{(0)}(z_j) - \sigma_{*j}\Phi^-_{(0)}(z_j)$ is a well-defined Fock space field,

$$\sum \sigma_j \Phi_{(0)}(z_1) + \sum_{j>1} \sigma_j \Phi^+_{(0)}(z_j, z_1) - \sigma_{*j}\Phi^-_{(0)}(z_j, z_1).$$

Suppose $\mathbf{E}\,[\mathcal{O}^{(\sigma,\sigma_*)}]$ is expressed as $M^{(\boldsymbol{\sigma},\boldsymbol{\sigma}_*)}$, $\widetilde{M}^{(\boldsymbol{\sigma},\boldsymbol{\sigma}_*)}$ in terms of two different conformal transformations $w, \widetilde{w}$ from (D, q_-, q_+) onto $(\mathbb{H}, -1, 1)$. Since a nontrivial element h in $\mathrm{Aut}(\mathbb{H}, -1, 1)$ is of the form

$$h(z) = \frac{az+1}{z+a}, \qquad (a \in \mathbb{R} \setminus [-1, 1]),$$

the ratio of $\widetilde{M}^{(\boldsymbol{\sigma},\boldsymbol{\sigma}_*)}$ and $M^{(\boldsymbol{\sigma},\boldsymbol{\sigma}_*)}$ is

$$\frac{(a^2-1)^{\sum_{j=1}^n(\sigma_j+\sigma_{*j})}}{\prod (w_j+a)^{\sigma_j^2+\sigma_j\sigma_{*j}+\sum_{k\neq j}\sigma_j(\sigma_k+\sigma_{*k})}(\bar{w}_j+a)^{\sigma_{*j}^2+\sigma_j\sigma_{*j}+\sum_{k\neq j}\sigma_{*j}(\sigma_k+\sigma_{*k})}}.$$

Thus the multi-vertex field $\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ with the neutrality condition is $\mathrm{Aut}(D, q_\pm)$-invariant.

Finally let us show that Ward's OPE holds for $\mathcal{O}$. Since the multi-vertex field is a differential, it is enough to verify Ward's OPE in the upper half-plane. In the identity chart of the upper half-plane,

$$J = J_{(0)} + ib\frac{2z}{1-z^2}, \quad T = T_{(0)} - jJ_{(0)} + ib\partial J_{(0)} + (ib\partial j - \frac{1}{2}j^2),$$

where $j = 2ibz/(1-z^2)$ and $T_{(0)} = -\frac{1}{2}J_{(0)} \odot J_{(0)}$. In the simplest case $b = 0$, let us show that the singular part of operator product expansion of $T_{(0)}(\zeta)$ and $\mathcal{O}(\boldsymbol{z})$ is

$$\frac{\sigma^2}{2}\frac{\mathcal{O}(\boldsymbol{z})}{(\zeta-z_j)^2} + i\sigma\frac{J_{(0)}(z_j)\odot\mathcal{O}(\boldsymbol{z})}{\zeta-z_j} + \Big(\frac{\sigma_j\sigma_{*j}}{z_j-\bar{z}_j} + \sum_{k\neq j}\frac{\sigma_j\sigma_k}{z_j-z_k} + \frac{\sigma_j\sigma_{*k}}{z_j-\bar{z}_k}\Big)\frac{\mathcal{O}(\boldsymbol{z})}{\zeta-z_j}$$

as $\zeta \to z_j$. For this, let

$$\begin{aligned} F \equiv F(\zeta, \boldsymbol{z}) &:= \mathbf{E}[J_{(0)}(\zeta)\sum\Big(i\sigma_j\Phi^+_{(0)}(z_j) - i\sigma_{*j}\Phi^-_{(0)}(z_j)\Big)] \\ &= \frac{1}{i}\sum\Big(\frac{\sigma_j}{\zeta-z_j} + \frac{\sigma_{*j}}{\zeta-\bar{z}_j}\Big). \end{aligned}$$

It follows from Wick's calculus that

$$T_{(0)}(\zeta)\,\mathcal{O}(\boldsymbol{z}) = T_{(0)}(\zeta)\odot\mathcal{O}(\boldsymbol{z}) - F\,J_{(0)}(\zeta)\odot\mathcal{O}(\boldsymbol{z}) - \frac{1}{2}\,F^2\,\mathcal{O}(\boldsymbol{z}).$$

While the first term $T_{(0)}(\zeta)\odot\mathcal{O}(\boldsymbol{z})$ has no contribution to Ward's OPE for $\mathcal{O}$, the second term $-F(\zeta,\boldsymbol{z})\,J_{(0)}(\zeta)\odot\mathcal{O}(\boldsymbol{z})$ has the singular part

$$i\sigma\frac{J_{(0)}(z_j)\odot\mathcal{O}(\boldsymbol{z})}{\zeta-z_j},\qquad (\zeta\to z_j).$$

The singular part of the last term $-\frac{1}{2}\,F(\zeta,\boldsymbol{z})^2\,\mathcal{O}(\boldsymbol{z})$ is

$$\frac{\sigma^2}{2}\frac{\mathcal{O}(\boldsymbol{z})}{(\zeta-z_j)^2} + \Big(\frac{\sigma_j\sigma_{*j}}{z_j-\bar z_j} + \sum_{k\neq j}\frac{\sigma_j\sigma_k}{z_j-z_k} + \frac{\sigma_j\sigma_{*k}}{z_j-\bar z_k}\Big)\frac{\mathcal{O}(\boldsymbol{z})}{\zeta-z_j},\qquad (\zeta\to z_j).$$

This proves Ward's OPE when $b=0$. For $b\neq 0$, we just need to show that

$$j(\zeta)J_{(0)}(\zeta)\,\mathcal{O}(\boldsymbol{z}) \sim 2\sigma_j b\frac{z_j}{1-z_j^2}\frac{\mathcal{O}(\boldsymbol{z})}{\zeta-z_j},\quad ib\partial J_{(0)}(\zeta)\,\mathcal{O}(z,z_0)\sim -\sigma_j b\,\frac{\mathcal{O}(\boldsymbol{z})}{(\zeta-z_j)^2}.$$

Both of the singular OPEs follow from

$$J_{(0)}(\zeta)\,\mathcal{O}(z,z_0)\sim -i\sigma_j\,\frac{\mathcal{O}(\boldsymbol{z})}{\zeta-z_j}.$$

Ward's OPE for $\bar{\mathcal{O}}$ can be obtained in a similar way. □

5.3 Coulomb gas correlation functions

In this section we represent $\mathbf{E}\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]$ in terms of Coulomb gas correlation functions $\mathcal{C}_{(b)}[\boldsymbol{\sigma}+b\cdot q_-+b\cdot q_+,\boldsymbol{\sigma}_*]$ and the *puncture operator* $\mathcal{P}_{(b)}$. Let $[\boldsymbol{\tau},\boldsymbol{\tau}_*]$ be a double divisor with the neutrality condition (NC_b):

$$\int\boldsymbol{\tau}+\boldsymbol{\tau}_* = 2b. \tag{5.3.1}$$

We define Coulomb gas correlation function (up to constant) $\mathcal{C}_{(b)}[\boldsymbol{\tau},\boldsymbol{\tau}_*]$ as a non-random field satisfying the following conditions:

- It is a (h_j,h_{*j})-differential with respect to z_j, where $z_j\in\operatorname{supp}\boldsymbol{\tau}\,\cup$

supp$\boldsymbol{\tau}_*$ and

$$(h_j, h_{*j}) = (\frac{\tau_j^2}{2} - \tau_j b, \frac{\tau_{*j}^2}{2} - \tau_{*j} b).$$

- Assume $\infty \notin \mathrm{supp}\boldsymbol{\tau} \cup \mathrm{supp}\boldsymbol{\tau}_*$. Then in the identity chart of $\mathbb{H}$, we define

$$[\mathcal{C}_{(b)} \| \mathrm{id}_{\mathbb{H}}][\boldsymbol{\tau}, \boldsymbol{\tau}_*] = \prod_{j<k} \Big((z_j - z_k)^{\tau_j \tau_k} (\overline{z_j} - z_k)^{\tau_{*j}\tau_k} \times (z_j - \overline{z_k})^{\tau_j \tau_{*k}} (\overline{z_j} - \overline{z_k})^{\tau_{*j}\tau_{*k}} \Big). \tag{5.3.2}$$

To be a well-defined non-random field, $\mathcal{C}_{(b)}[\boldsymbol{\tau}, \boldsymbol{\tau}_*]$ should be Möbius invariant on $\mathbb{H}$. Since (real) translation invariance of $\mathcal{C}_{(b)}$ is obvious, we need to check invariance under the dilation $\psi(z) = az$, $(a \in \mathbb{R})$ and the inversion $\psi(z) = -1/z$.

Lemma 5.3.1. *If $\boldsymbol{\tau}, \boldsymbol{\tau}_* : \mathbb{H} \cup \mathbb{R} \mapsto \mathbb{R}$ satisfy the neutrality condition (NC_b), and ψ is a Möbius map such that $\psi(z_j) \neq \infty$, then*

$$\mathcal{C}_{(b)}[\boldsymbol{\tau}, \boldsymbol{\tau}_*] = \mathcal{C}_{(b)}[\psi(\boldsymbol{\tau}), \psi(\boldsymbol{\tau}_*)] \prod_j (\psi'(z_j))^{h_j} (\overline{\psi'(z_j)})^{h_{*j}}, \tag{5.3.3}$$

where $\psi(\boldsymbol{\tau}) = \sum \tau_j \psi(z_j)$ and $\psi(\boldsymbol{\tau}_*) = \sum \tau_{*j} \psi(z_j)$.

Proof. For a dilation $\psi(z) = az$ with $a \in \mathbb{R}$, we have

$$\mathcal{C}_{(b)}[\psi(\boldsymbol{\tau}), \psi(\boldsymbol{\tau}_*)] = \mathcal{C}_{(b)}[\boldsymbol{\tau}, \boldsymbol{\tau}_*](a)^{\sum_{j<k} \tau_j\tau_k + \tau_{*j}\tau_k + \tau_j\tau_{*k} + \tau_{*j}\tau_{*k}},$$

and

$$\prod_j (\psi'(z_j))^{h_j} (\overline{\psi'(z_j)})^{h_{*j}} = (a)^{\sum_j h_j + h_{*j}}.$$

Note that, under the neutrality condition (NC_b), we have

$$\begin{aligned} &\sum_j h_j + h_{*j} + \sum_{j<k} \tau_j\tau_k + \tau_{*j}\tau_k + \tau_j\tau_{*k} + \tau_{*j}\tau_{*k} \\ &= \frac{1}{2}(\int \boldsymbol{\tau} + \boldsymbol{\tau}_*)^2 - b(\int \boldsymbol{\tau} + \boldsymbol{\tau}_*) = 0. \end{aligned}$$

Therefore, $\mathcal{C}_{(b)}$ is invariant under any dilation on $\mathbb{H}$. For the inversion $\psi(z) = -1/z$, we have

$$\frac{\mathcal{C}_{(b)}[\psi(\boldsymbol{\tau}),\psi(\boldsymbol{\tau}_*)]}{\mathcal{C}_{(b)}[\boldsymbol{\tau},\boldsymbol{\tau}_*]} = \prod_{j<k}(z_jz_k)^{-\tau_j\tau_k}(z_j\bar{z}_k)^{-\tau_j\tau_{*k}}(\bar{z}_jz_k)^{-\tau_{*j}\tau_k}(\bar{z}_j\bar{z}_k)^{-\tau_{*j}\tau_{*k}},$$

and

$$\prod_j(\psi'(z_j))^{h_j}(\overline{\psi'(z_j)})^{h_{*j}} = \prod_j(z_j)^{-2h_j}(\bar{z}_j)^{-2h_{*j}}.$$

Under the neutrality condition (NC_b), the exponents of $1/z_j$ in the right hand side of (5.3.3) is given by

$$2h_j + \tau_j\tau_{*j} + \tau_j\sum_{k\neq j}\Big(\tau_k + \tau_{*k}\Big) = (\tau_j^2 - 2b\tau_j) + \tau_j\tau_{*j} + \tau_j(2b - \tau_j - \tau_{*j}) = 0.$$

Similarly we can check that the exponents of $\bar{z}_j$ are zero. $\square$

Proposition 5.3.2. *Under the neutrality condition* (NC_b)*, the the differentials* $\mathcal{C}_{(b)}[\boldsymbol{\tau},\boldsymbol{\tau}_*]$ *are* $\mathrm{Aut}(\overline{D})$*-invariant.*

Proof. Since $\mathcal{C}_{(b)}$ are differentials, it is enough to prove in the case $D = \mathbb{H}$. Due to the previous lemma, it remains the case which the charge at ∞ (i.e., $\boldsymbol{\tau}(\infty)$) is not zero. By the translation invariance, we assume $\boldsymbol{\tau}(0) = \boldsymbol{\tau}_*(0) = 0$. For the inversion $\psi(z) = -1/z$, the conformal invariance of $\mathcal{C}_{(b)}[\boldsymbol{\tau},\boldsymbol{\tau}_*]$ under ψ means

$$\begin{aligned}(\mathcal{C}_{(b)}\|\mathrm{id})[\boldsymbol{\tau},\boldsymbol{\tau}_*] &= (\mathcal{C}_{(b)}\|\psi^{-1})[\boldsymbol{\tau},\boldsymbol{\tau}_*]\\ &= (\mathcal{C}_{(b)}\|\mathrm{id})[\psi(\boldsymbol{\tau}),\psi(\boldsymbol{\tau}_*)](\phi'(0))^{h_\infty}\prod_j(\psi'(z_j))^{h_j}(\psi'(\overline{z_j}))^{h_{*j}},\end{aligned}$$

where $\phi = \mathrm{id}$ is the transition map between the two charts at infinity. By the definition of $\mathcal{C}_{(b)}$,

$$\begin{aligned}\frac{(\mathcal{C}_{(b)}\|\mathrm{id})[\psi(\boldsymbol{\tau}),\psi(\boldsymbol{\tau}_*)]}{(\mathcal{C}_{(b)}\|\mathrm{id})[\boldsymbol{\tau},\boldsymbol{\tau}_*]} &= \prod_{j<k}(z_jz_k)^{-\tau_j\tau_k}(z_j\overline{z_k})^{-\tau_j\tau_{*k}}(\overline{z_j}z_k)^{-\tau_{*j}\tau_k}(\overline{z_j}\overline{z_k})^{-\tau_{*j}\tau_{*k}}\\ &\times\prod_j(z_j)^{-\tau_\infty\tau_j}(\overline{z_j})^{-\tau_\infty\tau_{*j}}.\end{aligned}$$

Therefore, inversion invariance of $\mathcal{C}$ is reduced from the identity

$$\prod_{j<k}(z_j z_k)^{\tau_j\tau_k}(z_j\overline{z_k})^{\tau_j\tau_{*k}}(\overline{z_j}z_k)^{\tau_{*j}\tau_k}(\overline{z_j z_k})^{\tau_{*j}\tau_{*k}}\prod_j(z_j)^{\tau_\infty\tau_j}(\overline{z_j})^{\tau_\infty\tau_{*j}}$$
$$=\prod_j(z_j)^{-2h_j}(\overline{z_j})^{-2h_{*j}}.$$

This holds by the neutrality condition (NC_b) because the exponent of z_j (resp. $\overline{z_j}$) on the left-hand side is

$$\tau_j^2 - 2\tau_j b + \tau_j(\tau_\infty + \sum_{k\neq j}\tau_k) = 0.$$

Finally, for a dilation $\psi(z) = az$ $(a > 0)$, we need to prove that

$$(\mathcal{C}_{(b)}\|\mathrm{id})[\boldsymbol{\tau},\boldsymbol{\tau}_*] = (\mathcal{C}_{(b)}\|\psi^{-1})[\boldsymbol{\tau},\boldsymbol{\tau}_*]$$
$$= (\mathcal{C}_{(b)}\|\mathrm{id})[\psi(\boldsymbol{\tau}),\psi(\boldsymbol{\tau}_*)](\phi'(0))^{h_\infty}\prod_j(\psi'(z_j))^{h_j}(\psi'(\overline{z_j}))^{h_{*j}},$$

where $\phi(w) = w/a$ is the transition map between the two charts at infinity. From the definition of $\mathcal{C}_{(b)}$, we have

$$(\mathcal{C}_{(b)}\|\mathrm{id})[\psi(\boldsymbol{\tau}),\psi(\boldsymbol{\tau}_*)] = (\mathcal{C}_{(b)}\|\mathrm{id})[\boldsymbol{\tau},\boldsymbol{\tau}_*](a)^{\sum_{j<k}\tau_j\tau_k+\tau_j\tau_{*k}+\tau_{*j}\tau_k+\tau_{*j}\tau_{*k}}.$$

Thus the conformal invariance is induced from the identity

$$\sum_j\Big(h_j + h_{*j}\Big) + \sum_{j<k}\Big(\tau_j\tau_k + \tau_j\tau_{*k} + \tau_{*j}\tau_k + \tau_{*j}\tau_{*k}\Big) = h_\infty.$$

It follows from the neutrality condition (NC_b) that

$$\sum_j\Big(h_j + h_{*j}\Big) + \sum_{j<k}\Big(\tau_j\tau_k + \tau_j\tau_{*k} + \tau_{*j}\tau_k + \tau_{*j}\tau_{*k}\Big)$$
$$= (\sum_j\tau_j + \tau_{*j})^2/2 - (\sum_j\tau_j + \tau_{*j})b = (2b-\tau_\infty)^2/2 - (2b-\tau_\infty)b = h_\infty.$$

$\square$

Let us define the dipolar "boundary puncture operator" $\mathcal{P}_{(b)}$ as

$$\mathcal{P}_{(b)}(q_\pm) := \mathcal{C}_{(b)}[b\cdot q_- + b\cdot q_+, 0].$$

By the definition, $\mathcal{P}_{(b)} = 1$ in the identity chart of $\mathbb{H}$ and a boundary differential with conformal dimension $-\frac{1}{2}b^2$ with respect to q_- and q_+. Now we represent multi-vertex fields in terms of the Coulomb gas correlation functions.

Proposition 5.3.3. *If a double divisor* $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*)$ *satisfies the neutrality condition* (NC_0)*, then*

$$\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] = \mathcal{P}_{(b)}^{-1}\mathcal{C}_{(b)}[\boldsymbol{\sigma} + b \cdot q_- + b \cdot q_+, \boldsymbol{\sigma}_*]\mathcal{V}^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*], \tag{5.3.4}$$

where $\mathcal{V}^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] = e^{\odot i\Phi_{(0)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]}$.

Proof. Since both side of (5.3.4) have same conformal dimensions at each nodes of $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*)$ and $q_\pm$, we only need to check in the identity chart of $\mathbb{H}$. It is directly induced from the definitions of $\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ and $\mathcal{C}_{(b)}$. □

Remark. In the chordal case, we took boundary puncture operator $\mathcal{P}_{(b)}$ as $\mathcal{P}_{(b)}(q) := \mathcal{C}_{(b)}[2b \cdot q, 0]$ where q is a marked boundary point. Remark that $\mathcal{C}_{(b)}[2b \cdot q, 0]$ is a constant field and $h_q = 0$. For example, in [12, Lecture 12.2], chiral vertex field $V^{i\sigma}(z, z_0)$ is given by (here, we use $i\sigma$ instead of α)

$$\begin{aligned} V^{i\sigma}(z, z_0) &= w'(z)^{\sigma^2/2-\sigma b}w'(z_0)^{\sigma^2/2+\sigma b}(w(z) - w(z_0))^{-\sigma^2}e^{\odot i\sigma\Phi^+_{(0)}(z,z_0)} \\ &= \mathcal{P}_{(b)}^{-1}\mathcal{C}_{(b)}[\sigma \cdot z - \sigma \cdot z_0 + 2b \cdot q, 0]e^{\odot i\sigma\Phi^+_{(0)}(z,z_0)}, \end{aligned}$$

where w is any conformal map $(D, q) \to (\mathbb{H}, \infty)$.

Also, in the radial case, the puncture operator $\mathcal{P}_{(b)}$ is given by $\mathcal{C}_{(b)}[b{\cdot}q, b{\cdot}q]$ where q is a marked interior point. As we discussed before, we have

$$[\mathcal{C}_{(b)} \| \mathrm{id}_{\mathbb{D}}][\boldsymbol{\tau}, \boldsymbol{\tau}_*] = \prod_{j<k}(z_j - z_k)^{\tau_j\tau_k}(1 - z_j\overline{z_k})^{\tau_j\tau_{*k}}(1 - \overline{z_j}z_k)^{\tau_{*j}\tau_k}(\overline{z_j} - \overline{z_k})^{\tau_{*j}\tau_{*k}}.$$

Remark that $\mathcal{C}_{(b)}[b \cdot q, b \cdot q]$ is a $(-\frac{1}{2}b^2, -\frac{1}{2}b^2)$-differential at q and constant in the identity chart of $\mathbb{D}$. Similarly as Proposition 5.3.3, we can prove that, under the neutrality condition,

$$\mathcal{O}^{radial}[\boldsymbol{\tau}, \boldsymbol{\tau}_*] = \frac{\mathcal{C}_{(b)}[\boldsymbol{\tau} + b \cdot q, \boldsymbol{\tau}_* + b \cdot q]}{\mathcal{C}_{(b)}[b \cdot q, b \cdot q]}e^{\odot i\Phi[\boldsymbol{\tau}, \boldsymbol{\tau}_*]},$$

where $\mathcal{O}^{radial}[\boldsymbol{\tau}, \boldsymbol{\tau}_*]$ is defined in [11, Section 3.4].

We define a product of vertex fields as the normalized tensor product of them. If two vertex fields have no common nodes, then it is obvious that

$$\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\star\mathcal{O}[\boldsymbol{\tau},\boldsymbol{\tau}_*]=\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\mathcal{O}[\boldsymbol{\tau},\boldsymbol{\tau}_*]=\mathcal{O}[\boldsymbol{\sigma}+\boldsymbol{\tau},\boldsymbol{\sigma}_*+\boldsymbol{\tau}_*]$$

holds for disjoint double divisors $(\boldsymbol{\sigma},\boldsymbol{\sigma}_*)$ and $(\boldsymbol{\tau},\boldsymbol{\tau}_*)$. We have same formula even if two double divisor have common nodes.

Proposition 5.3.4. *If both $(\boldsymbol{\sigma},\boldsymbol{\sigma}_*)$ and $(\boldsymbol{\tau},\boldsymbol{\tau}_*)$ satisfy the neutrality condition (NC_0), then*

$$\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]\star\mathcal{O}[\boldsymbol{\tau},\boldsymbol{\tau}_*]=\mathcal{O}[\boldsymbol{\sigma}+\boldsymbol{\tau},\boldsymbol{\sigma}_*+\boldsymbol{\tau}_*].$$

Proof. It is enough to prove it in the holomorphic case $\boldsymbol{\sigma}_*=\boldsymbol{\tau}_*=\mathbf{0}$. We have the OPE of two formal Wick exponential fields as

$$\begin{aligned}e^{\odot i\sigma\Phi^+(\zeta)}e^{\odot i\tau\Phi^+(z)}&=e^{-\sigma\tau\mathbf{E}[\Phi^+(\zeta)\Phi^+(z)]}e^{\odot i\sigma\Phi^+(\zeta)+i\tau\Phi^+(z)}\\&=(\zeta-z)^{\sigma\tau}(w_z')^{\sigma\tau}e^{\odot i(\sigma+\tau)\Phi^+(z)}+o(1)\quad\text{as}\quad\zeta\to z.\end{aligned}$$

Hence, conformal dimension of $\mathcal{O}[\boldsymbol{\sigma}]*\mathcal{O}[\boldsymbol{\tau}]$ at $z_j\in\operatorname{supp}\boldsymbol{\sigma}\cup\operatorname{supp}\boldsymbol{\tau}$ is given by

$$\frac{\sigma_j^2}{2}-\sigma_jb+\frac{\tau_j^2}{2}-\tau_jb+\sigma_j\tau_j=\frac{(\sigma_j+\tau_j)^2}{2}-(\sigma_j+\tau_j)b.$$

Also, the both fields are scalar fields with respect to $q_\pm$. Therefore, it is enough to check in the identity chart of $\mathbb{H}$, which is directly following from the definition of the Coulomb gas correlation function. $\square$

5.4 Rooted multi-vertex fields

In Section 5.2, we defined multi-vertex fields as the OPE exponential of bosonic fields $\Phi_{(b)}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*]$ without charges at the marked boundary points q_- and q_+. Now, we extend our definition of vertex fields in a case such that $\sigma_-\equiv\boldsymbol{\sigma}(q_-)\neq0$ or $\sigma_+\equiv\boldsymbol{\sigma}(q_+)\neq0$. Let $(\boldsymbol{\sigma},\boldsymbol{\sigma}_*;\sigma_-,\sigma_+)$ denote the double divisor $(\boldsymbol{\sigma}+\sigma_-\cdot q_-+\sigma_+\cdot q_+,\boldsymbol{\sigma}_*)$. Recall that

$$\Phi_{(b)}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*;\sigma_-,\sigma_+]=\Phi_{(0)}[\boldsymbol{\sigma},\boldsymbol{\sigma}_*;\sigma_-,\sigma_+]+ib\sum_j\sigma_j\log\frac{w_{z_j}'}{1-w_{z_j}^2}-\sigma_{*j}\frac{\bar w_{z_j}'}{1-\bar w_{z_j}^2}.$$

Proposition 5.4.1. *If a double divisor* $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+)$ *satisfies the neutrality condition* (NC_0)*, then*

$$\begin{aligned}\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+] &\equiv e^{*i\Phi_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+]} \\ &= e^{-c[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+]} e^{i\mathbf{E}\Phi_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+]} \mathcal{V}^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+],\end{aligned}$$

where $\mathcal{V}^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+] = e^{\odot i\Phi_{(0)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+]}$*. In particular,* $\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*, \sigma_-, \sigma_+]$ *belongs to* $\mathcal{F}_{(b)}$ *and is a* (h_j, h_{*j})*-differential with respect to* z_j*. Here*

$$(h_j, h_{*j}) = (\frac{\sigma_j^2}{2} - \sigma_j b, \frac{\sigma_{*j}^2}{2} - \sigma_{*j} b).$$

Also, it is a $(\sigma_\pm^2/2, 0)$*-differential with respect to* $q_\pm$*. More precisely, in the* $(\mathbb{H}, -1, 1)$*-uniformization with conformal transformation* $w : (D, q_-, q_+) \mapsto (\mathbb{H}, -1, 1)$*, we have*

$$\mathbf{E}\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+] = (w'_-)^{\sigma_-^2/2} (w'_+)^{\sigma_+^2/2} \prod_{j=1}^{n} M^{(\sigma_j, \sigma_{*j}; \sigma_-, \sigma_+)}(z_j) \prod_{j<k} I_{j,k}(z_j, z_k),$$

where $w'_- = w'(q_-), w'_+ = w'(q_+)$ *and*

$$\begin{aligned}M^{(\sigma, \sigma_*; \sigma_-, \sigma_+)}(z) = (w'(z))^{\sigma^2/2 - \sigma b} (\overline{w}'(z))^{\sigma_*^2/2 - \sigma_* b} (1 - w(z))^{\sigma(b+\sigma_+)} \\ \times (1 + w(z))^{\sigma(b+\sigma_-)} (1 - \overline{w}(z))^{\sigma_*(b+\sigma_+)} (1 + \overline{w}(z))^{\sigma_*(b+\sigma_-)},\end{aligned} \tag{5.4.1}$$

and $I_{j,k}$ *is same as* (5.2.2)*.*

Remark. We defined central charge modified field $\Phi_{(b)}$ at the marked boundary points $q_\pm$ as

$$\begin{aligned}\Phi^+_{(b)}(q_\pm) &:= \Phi^+ * 1(q_\pm) = \Phi^+_{(0)}(q_\pm), \\ \Phi^-_{(b)}(q_\pm) &:= \Phi^- * 1(q_\pm) = \Phi^-_{(0)}(q_\pm).\end{aligned}$$

Applying these to the OPE exponentials, we get

$$\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+] = \mathcal{O}[\boldsymbol{\sigma} + \sigma_- \cdot \zeta_- + \sigma_+ \cdot \zeta_+, \boldsymbol{\sigma}_*] * 1_{q_\pm},$$

where the OPE means that $\zeta_\pm \mapsto q_\pm$ in the given chart. For example

$$\mathcal{O}^{(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+)}(\boldsymbol{z}) := \lim_{\varepsilon \to 0} \frac{\mathcal{O}^{(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*)}(z) \mathcal{O}^{(\sigma_-)}(\zeta_\varepsilon^-) \mathcal{O}^{(\sigma_+)}(\zeta_\varepsilon^+)}{(\zeta_\varepsilon^- - q_-)^{\sigma_- b} (\zeta_\varepsilon^+ - q_+)^{\sigma_+ b}}, \tag{5.4.2}$$

where $\zeta_\varepsilon^\pm$ is at distance ε from $q_\pm$ in a given chart. We call this normalized limiting algorithm as *the rooting procedure.*

Example. For a real constant a, we define

$$\Psi(z) \equiv \mathcal{O}[a \cdot z; -\frac{a}{2}, -\frac{a}{2}].$$

By above proposition, we can write

$$\Psi(z) = (w'_-)^{\frac{1}{8}a^2}(w'_+)^{\frac{1}{8}a^2}\Big(\frac{w'(z)}{1-w(z)^2}\Big)^{h_{1,2}} e^{\odot\frac{1}{2}ia(\Phi^+(z,q_-)+\Phi^+(z,q_+))},$$

where $h_{1,2} := a^2/2 - ab$ and w is a conformal transformation from (D, q_-, q_+) onto $(\mathbb{H}, -1, 1)$.

We now extend the OPE family $\mathcal{F}_{(b)}$ to include rooted multi-vertex fields. This extension is natural in the sense that Ward's OPEs for multi-vertex fields survive under OPE product (or the rooting procedure.)

Proposition 5.4.2. *For a rooted multi-vertex field $\mathcal{O} \equiv \mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+]$, Ward's OPE*

$$T(\zeta)\mathcal{O}(\boldsymbol{z}) \sim h_j \frac{\mathcal{O}(\boldsymbol{z})}{(\zeta - z_j)^2} + \frac{\partial_{z_j}\mathcal{O}(\boldsymbol{z})}{\zeta - z_j}, \qquad (\zeta \to z_j),$$

holds and similar equation holds (with $\bar{h}_{*j}$) for $\bar{\mathcal{O}}$.

We also represent rooted multi-vertex fields in terms of Coulomb gas correlation functions.

Proposition 5.4.3. *If a double divisor $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+)$ satisfies the neutrality condition (NC_0), then*

$$\begin{aligned}\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+] =& \mathcal{P}_{(b)}^{-1}\mathcal{C}_{(b)}[\boldsymbol{\sigma} + (b+\sigma_-)\cdot q_- + (b+\sigma_+)\cdot q_+, \boldsymbol{\sigma}_*] \\ &\times \mathcal{V}^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+],\end{aligned} \tag{5.4.3}$$

where $\mathcal{V}^{\odot}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+] = e^{\odot i\Phi_{(0)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+]}$.

Proof. The proof is essentially same as the proof of Proposition 5.3.3. We only need to check conformal dimensions at $q_\pm$. Note that the conformal dimension of the right-hand side is given by

$$[\frac{(b+\sigma_+)^2}{2} - (b+\sigma_+)b] + \frac{b^2}{2} = \frac{\sigma_+^2}{2}.$$

□

Finally we define the product of two rooted multi-vertex fields as their normalized tensor product. Using the same argument in Proposition 5.3.4 and easy dimension calculus at the marked boundary points $q_\pm$, we have the following proposition.

Proposition 5.4.4. *If both* $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+)$ *and* $(\boldsymbol{\tau}, \boldsymbol{\tau}_*; \tau_-, \tau_+)$ *satisfy the neutrality condition* (NC_0), *then*

$$\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+] \star \mathcal{O}[\boldsymbol{\tau}, \boldsymbol{\tau}_*; \tau_-, \tau_+] = \mathcal{O}[\boldsymbol{\sigma}+\boldsymbol{\tau}, \boldsymbol{\sigma}_*+\boldsymbol{\tau}_*; \sigma_-+\tau_-, \sigma_++\tau_+].$$

5.5 Ward's identity and equation for rooted multi-vertex fields

Applying the rooting procedure, we derive the following Ward's identity for rooted multi-vertex fields.

Proposition 5.5.1. *For a rooted vertex field* $\mathcal{O}(\boldsymbol{z}) \equiv \mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+]$ *with the neutrality condition* (NC_0), *we have Ward's identity*

$$\mathbf{E}\,[\,W(v; \bar{D}\setminus\{q_\pm\})\,\mathcal{O}(\boldsymbol{z})\,] = \mathbf{E}\,[\,\mathcal{L}_v\,\mathcal{O}(\boldsymbol{z})\,] + (h_- + h_+)\,\mathbf{E}[\,\mathcal{O}(\boldsymbol{z})\,],\ (h_\pm = \frac{1}{2}\sigma_\pm^2), \tag{5.5.1}$$

where v is a non-random local holomorphic vector field with $v(q_\pm) = 0, v'(q_\pm) = 1$, and $z_j \in D_{\mathrm{hol}}(v)$. The Lie derivative operator $\mathcal{L}_v$ does not apply to the points $q_\pm$.

Proof. Since vertex fields are differentials, it suffices to perform the computation in the $(\mathbb{H}, -1, 1)$-uniformization. Suppose that

$$\mathcal{O}(\boldsymbol{z}) = \lim_{\varepsilon\to 0} \frac{\mathcal{O}^{(\boldsymbol{\sigma},\boldsymbol{\sigma}_*)}(\boldsymbol{z})\mathcal{O}^{(\sigma_-)}(\zeta_\varepsilon^-)\mathcal{O}^{(\sigma_+)}(\zeta_\varepsilon^+)}{(1+\zeta_\varepsilon^-)^{\mu_-}(1-\zeta_\varepsilon^+)^{\mu_+}},$$

where $\zeta_\varepsilon^\pm$ is at distance ε from ± 1 and $\mu_\pm = \sigma_\pm b$. We write $\mathcal{O}(\boldsymbol{z}, \zeta_-, \zeta_+)$ for the (unrooted) multi-vertex field $\mathcal{O}^{(\boldsymbol{\sigma},\boldsymbol{\sigma}_*)}(\boldsymbol{z})\mathcal{O}^{(\sigma_-)}(\zeta_-)\mathcal{O}^{(\sigma_+)}(\zeta_+)$.

Since Ward's identity holds for $\mathcal{O}(\boldsymbol{z}, \zeta_-, \zeta_+)$, it is enough to show that

$$\lim_{\varepsilon\to 0} \frac{\mathbf{E}\,[\mathcal{L}_v\,\mathcal{O}(\boldsymbol{z}, \zeta_\varepsilon^-, \zeta_\varepsilon^+)]}{(1+\zeta_\varepsilon^-)^{\mu_-}(1-\zeta_\varepsilon^+)^{\mu_+}} = \mathbf{E}\,[\mathcal{L}_v\,\mathcal{O}(\boldsymbol{z})] + (h_- + h_+)\,\mathbf{E}\,[\mathcal{O}(\boldsymbol{z})].$$

Clearly,

$$\lim_{\varepsilon\to 0}\frac{\mathbf{E}\,[\mathcal{L}_v(\boldsymbol{z})\,\mathcal{O}(\boldsymbol{z},\zeta_\varepsilon^-,\zeta_\varepsilon^+)]}{(1+\zeta_\varepsilon^-)^{\mu_-}(1-\zeta_\varepsilon^+)^{\mu_+}}=\mathbf{E}\,[\mathcal{L}_v\,\mathcal{O}(\boldsymbol{z})],$$

where $\mathcal{L}_v(\boldsymbol{z})=\sum(v(z_j)\partial_j+\overline{v(z_j)}\bar\partial_j+h_jv'(z_j)+h_{*j}\overline{v'(z_j)})$. Write $\mathcal{L}_v(\zeta_\pm)$ for the differential operator $\sum(v(\zeta_\pm)\partial_{\zeta_\pm}+\lambda_\pm v'(\zeta_\pm))$, where $\lambda_\pm=\sigma_\pm^2/2-\sigma_\pm b$. Then

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{\mathbf{E}\,[\mathcal{L}_v(\zeta_\pm)\,\mathcal{O}(\boldsymbol{z},\zeta_\varepsilon^-,\zeta_\varepsilon^+)]}{(1+\zeta_\varepsilon^-)^{\sigma_-b}(1-\zeta_\varepsilon^+)^{\sigma_+b}}\\&=\lim_{\varepsilon\to 0}(\mu_\pm\frac{v(\zeta_\varepsilon^\pm)}{\zeta_\varepsilon^\pm\mp 1}+\lambda_\pm v'(\zeta_\varepsilon^\pm))\frac{\mathbf{E}\,[\mathcal{O}(\boldsymbol{z},\zeta_\varepsilon^-,\zeta_\varepsilon^+)]}{(1+\zeta_\varepsilon^-)^{\mu_-}(1-\zeta_\varepsilon^+)^{\mu_+}}\\&=h_\pm\,\mathbf{E}\,[\mathcal{O}(\boldsymbol{z})],\end{aligned}$$

which completes the proof. $\square$

Since the dipolar Loewner vector field

$$v_\zeta(z)=\frac{1-z^2}{2}\frac{1-\zeta z}{\zeta-z}$$

in the upper half-plane satisfies $v_\zeta(\pm1)=0, v_\zeta'(\pm1)=1$, we can apply the previous proposition to the vector field v_ζ together with Proposition 4.3.1 and derive the following form of Ward's equation for a rooted multi-vertex field.

Proposition 5.5.2. *For a rooted multi-vertex field $\mathcal{O}\equiv\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma_*};\sigma_-,\sigma_+]$ in the extended OPE family $\mathcal{F}_{(b)}$, we have*

$$\frac{(1-\zeta^2)^2}{2}\mathbf{E}[T_{(b)}(\zeta)\mathcal{O}]=\mathbf{E}[(\mathcal{L}^+_{v_\zeta}+\mathcal{L}^-_{v_{\bar\zeta}})\mathcal{O}]+(h_-+h_+-b^2)\mathbf{E}[\mathcal{O}],$$

where $T_{(b)}$ and $\mathcal{O}$ are evaluated in the identity chart of the upper half-plane.

We now generalize the previous proposition.

Proposition 5.5.3. *For a 1-point rooted vertex field V, and a rooted multi-*

vertex field $\mathcal{O}$ in $\mathcal{F}_{(b)}$, in the identity chart of the upper half-plane, we have

$$\begin{aligned}&\mathbf{E}\,[V(z)\star\mathcal{L}^+_{v_z}\mathcal{O}]+\mathbf{E}\,[\mathcal{L}^-_{v_{\bar z}}(V(z)\star\mathcal{O})]\\&\qquad=\frac{(1-z^2)^2}{2}\,\mathbf{E}\,[(L_{-2}V)(z)\star\mathcal{O}]-\frac{3z(1-z^2)}{2}\,\mathbf{E}\,[(L_{-1}V(z))\star\mathcal{O}]\\&\qquad+(\frac{3z^2-1}{2}\,h(V)+b^2-(h_-(V\star\mathcal{O})+h_+(V\star\mathcal{O})))\,\mathbf{E}\,[V(z)\star\mathcal{O}],\end{aligned}$$

where $h(V)$ is the conformal dimension of V with respect to z and $h_\pm(V\star\mathcal{O})$ is the boundary dimension of $V\star\mathcal{O}$ with respect to $q_\pm$.

Remark. The Lie derivative operators $\mathcal{L}^\pm_v$ in the above proposition do not apply to the points $q_\pm$. For the definition of a field $V\star\mathcal{L}^+_v\mathcal{O}$, the rooting rules can be applied to the tensor product of a bi-vertex field and the Lie derivative of a multi-vertex field.

Proof. Since Leibniz's rule applies to $\star$-products, i.e.,

$$\mathcal{L}^+_v(V\star\mathcal{O})=(\mathcal{L}^+_vV)\star\mathcal{O}+V\star(\mathcal{L}^+_v\mathcal{O}),$$

it follows from Proposition 5.5.2 that

$$\begin{aligned}&\mathbf{E}[V(z)\star\mathcal{L}^+_{v_\zeta}\mathcal{O}]+\mathbf{E}\,[\mathcal{L}^-_{v_{\bar\zeta}}(V(z)\star\mathcal{O})]\\&=\frac{(1-\zeta^2)^2}{2}\,\mathbf{E}\,[T(\zeta)V(z)\star\mathcal{O}]-\mathbf{E}\,[(\mathcal{L}^+_{v_\zeta}V)(z)\star\mathcal{O}]\\&\quad-(h_-+h_+-b^2)\,\mathbf{E}\,[V(z)\star\mathcal{O}],\end{aligned}$$

where $h_\pm=h_\pm(V\star\mathcal{O})$. By (4.3.1),

$$\begin{aligned}&\lim_{\zeta\to z}\frac{(1-\zeta^2)^2}{2}\,T(\zeta)V(z)-\mathcal{L}^+_{v_\zeta}V(z)\\&\qquad=\frac{(1-z^2)^2}{2}\,T*V(z)-\frac{3z(1-z^2)}{2}\,T*_{-1}V(z)+\frac{3z^2-1}{2}\,T*_{-2}V(z).\end{aligned}$$

Since V is a primary field in $\mathcal{F}_{(b)}$ with conformal dimension $[h,0]$, $L_0V=hV$, see Proposition 3.7.1.

$\square$

5.6 Level two degeneracy equations

In Section 2.2 we introduce insertion fields

$$e^{\odot\frac{1}{2}ia(\Phi^+(p,q_-)+\Phi^+(p,q_+))} \qquad (p \in \partial D \setminus \bar{Q}).$$

These Wick's exponentials can be normalized properly such that the normalized fields form a one-parameter family of $\mathrm{Aut}(D, q_-, q_+)$-invariant (Virasoro) primary fields

$$\Psi := \mathcal{O}[a \cdot z, 0; -\frac{1}{2}a, -\frac{1}{2}a]$$

in the extended OPE family $\mathcal{F}_{(b)}$. By definition of rooted vertex fields,

$$\Psi(z) = (w'_-)^{\frac{1}{8}a^2}(w'_+)^{\frac{1}{8}a^2}\Big(\frac{w'(z)}{1-w(z)^2}\Big)^{h_{1,2}} e^{\odot\frac{1}{2}ia(\Phi^+(z,q_-)+\Phi^+(z,q_+))}, \quad (5.6.1)$$

where $h_{1,2} = \frac{1}{2}a^2 - ab$, and w is a conformal transformation from (D, q_-, q_+) onto $(\mathbb{H}, -1, 1)$. (Recall that $w'_\pm = w'(q_\pm)$.) As rooted vertex fields, Ψ are current primary fields (see Section 3.7) with charges $q = a, q_* = 0$, i.e.,

$$J_0\Psi = -ia\Psi, \quad J_0\bar{\Psi} = 0, \quad J_n\Psi = J_n\bar{\Psi} = 0\,(n \geq 1), \qquad (5.6.2)$$

where the modes J_n of the current field are defined as

$$J_n(z) := \frac{1}{2\pi i}\oint_{(z)} (\zeta - z)^n J(\zeta)\,d\zeta. \qquad (5.6.3)$$

To show (5.6.2), one needs to check the singular OPEs

$$J_{(0)}(\zeta)\Psi(z) \sim -ia\frac{\Psi(z)}{\zeta - z}, \qquad J_{(0)}(\zeta)\overline{\Psi(z)} \sim 0$$

in the identity chart of the upper half-plane. The first singular OPE follows from Wick's calculus

$$J_{(0)}(\zeta)\Psi(z) = J_{(0)}(\zeta) \odot \Psi(z) + \frac{ia}{2}\,\mathbf{E}[J_{(0)}(\zeta)(\Phi^+_{(0)}(z,-1) + \Phi^+_{(0)}(z,1))]\,\Psi(z)$$

and $\mathbf{E}[J_{(0)}(\zeta)(\Phi^+_{(0)}(z,-1) + \Phi^+_{(0)}(z,1))] = -2/(\zeta - z) + 2\zeta/(\zeta^2 - 1)$. The second OPE follows from the similar Wick's decomposition and the fact that the correlation

$$\mathbf{E}[J_{(0)}(\zeta)(\Phi^-_{(0)}(z,-1) + \Phi^-_{(0)}(z,1))] = -\frac{2}{\zeta - \bar{z}} + \frac{2\zeta}{\zeta^2 - 1}$$

has no singular term.

Proposition 5.6.1. *If* $2a(a+b) = 1,$ *then*

$$T_{(b)} * \Psi = \frac{1}{2a^2}\partial^2\Psi.$$

This proposition follows immediately from the characterization of level two degenerate current primary fields, see Proposition 3.7.2. We combine Proposition 5.6.1 with Ward's equations to prove Theorems 2.2.1 and 2.3.1.

Chapter 6

Connection between dipolar SLE and CFT

After we introduce the insertion fields Ψ as boundary condition changing operators acting on Fock space functionals/fields, we prove that correlation functions of fields in the OPE family $\mathcal{F}_{(b)}$ of $\Phi_{(b)}$ under the insertion of $\Psi(p)/\mathbf{E}[\Psi(p)]$ are dipolar SLE$_\kappa(p \to Q)$ martingale-observables. The main ingredient for its proof is the BPZ-Cardy equation, which is derived from the level two degeneracy equation for Ψ and the Ward equation. As applications, we discuss the restriction property of dipolar SLE$_{8/3}$ and the dipolar version of the Friedrich-Werner's formula.

6.1 Boundary condition changing operator

We define a boundary condition changing operator $\mathcal{X} \mapsto \widehat{\mathcal{X}}$ as a linear operator acting on Fock space functionals in the following way. By definition, $\mathcal{X} \mapsto \widehat{\mathcal{X}}$ is given by the rules

$$\partial\mathcal{X} \mapsto \partial\widehat{\mathcal{X}}, \quad \bar{\partial}\mathcal{X} \mapsto \bar{\partial}\widehat{\mathcal{X}}, \quad \mathcal{X} \odot \mathcal{Y} \mapsto \widehat{\mathcal{X}} \odot \widehat{\mathcal{Y}},$$

and the formula

$$\sum \sigma_j \Phi^+_{(0)}(z_j) - \sigma_{*j}\Phi^-_{(0)}(z_j) \mapsto \sum -\frac{i\sigma_j a}{2}\log\frac{w_j^2}{1-w_j^2} + \frac{i\sigma_{*j}a}{2}\log\frac{\bar{w}_j^2}{1-\bar{w}_j^2} \\ + \sum \sigma_j \Phi^+_{(0)}(z_j) - \sigma_{*j}\Phi^-_{(0)}(z_j),$$

where w is the conformal transformation from (D, p, q_-, q_+) onto $(\mathbb{H}, 0, -1, 1)$ and the neutrality condition $\int \boldsymbol{\sigma} + \boldsymbol{\sigma}_* = 0$ holds.

Let us denote by $\widehat{\mathcal{F}}_{(b)}$ the image of $\mathcal{F}_{(b)}$ under this boundary condition changing operator $\mathcal{X} \mapsto \widehat{\mathcal{X}}$. Also we denote

$$\widehat{\mathbf{E}}[\mathcal{X}] := \frac{\mathbf{E}[\Psi(p)\mathcal{X}]}{\mathbf{E}[\Psi(p)]} = \mathbf{E}[e^{\odot \frac{1}{2} ia(\Phi^+(p,q_-)+\Phi^+(p,q_+))}\mathcal{X}].$$

As in the chordal case ([12, Proposition 14.1]), we have

$$\widehat{\mathbf{E}}[\mathcal{X}] = \mathbf{E}[\widehat{\mathcal{X}}], \tag{6.1.1}$$

where $\mathcal{X}$ is the string in $\mathcal{F}_{(b)}$ with nodes in $\bar{D} \setminus \{q_\pm\}$.

Examples. Let w be the conformal transformation from $(D, p, q_\pm)$ onto $(\mathbb{H}, 0, \pm 1)$.

(a) The bosonic field $\widehat{\Phi}$ is a real part of pre-pre-Schwarzian form of order ib,

$$\widehat{\Phi} = \Phi + a \arg \frac{w^2}{1-w^2} = \Phi_{(0)} + a \arg \frac{w^2}{1-w^2} - 2b \arg \frac{w'}{1-w^2};$$

(b) The current field $\widehat{J}$ is a pre-Schwarzian form of order ib,

$$\widehat{J} = J - ia\frac{w'}{w(1-w^2)} = J_{(0)} - ia\frac{w'}{w(1-w^2)} + ib(\frac{w''}{w'} + \frac{2ww'}{1-w^2});$$

(c) The Virasoro field $\widehat{T}$ is a Schwarzian form of order $\frac{1}{12}c$,

$$\widehat{T} = A_{(0)} - \hat{\jmath} J_{(0)} + ib\partial J_{(0)} + \frac{c}{12}S_w + h_{1,2}\frac{w'^2}{w^2(1-w^2)} + 4h_{0,1/2}(\frac{w'}{1-w^2})^2,$$

where $A_{(0)} = -\frac{1}{2}J_{(0)}\odot J_{(0)}$, $\hat{\jmath} = \mathbf{E}\,[\widehat{J}\,]$ and $h_{1,2} = \frac{1}{2}a^2 - ab$, $h_{0,1/2} = \frac{1}{8}a^2 - \frac{1}{2}b^2$;
(d) The multi-vertex field $\widehat{\mathcal{O}}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}]$ is a $[\boldsymbol{h}, \boldsymbol{h}_*]$-differential ($h_j = \frac{1}{2}\sigma_j^2 - \sigma_j b$, $h_{*j} = \frac{1}{2}\sigma_{*j}^2 - \sigma_{*j} b$),

$$\widehat{\mathcal{O}}^{(\boldsymbol{\sigma},\boldsymbol{\sigma_*})}(\boldsymbol{z}) = \prod \widehat{M}^{(\sigma_j,\sigma_{*j})}(z_j) \prod_{j<k} I_{j,k}\, e^{i\odot\sum \sigma_j \Phi^+_{(0)}(z_j) - \sigma_{*j}\Phi^-_{(0)}(z_j)},$$

where $\widehat{M}^{(\sigma_j,\sigma_{*j})}(z_j) = (w'_j)^{h_j}(\overline{w'}_j)^{h_{*j}}(1-w_j^2)^{\widehat{\mu}_j}(1-\bar{w}_j^2)^{\widehat{\mu}_{*j}} w_j^{\sigma_j a}\bar{w}_j^{\sigma_{*j} a}(w_j -$

$\bar{w}_j)^{\sigma\sigma_*}$ and interaction term $I_{j,k}$ is given by

$$I_{j,k} = (w_j - w_k)^{\sigma_j\sigma_k}(w_j - \bar{w}_k)^{\sigma_j\sigma_{*k}}(\bar{w}_j - w_k)^{\sigma_{*j}\sigma_k}(\bar{w}_j - \bar{w}_k)^{\sigma_{*j}\sigma_{*k}}.$$

The exponents are given by

$$\widehat{\mu}_j = \mu_j - \frac{1}{2}\sigma_j a = \sigma_j\Big(b - \frac{1}{2}a\Big), \quad \widehat{\mu}_{*j} = \mu_{*j} - \frac{1}{2}\sigma_{*j}a = \sigma_{*j}\Big(b - \frac{1}{2}a\Big).$$

Using the Coulomb gas correlation functions and (6.1.1), we can easily compute correlation function of $\widehat{\mathcal{O}}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}]$. Precisely, we have

$$\begin{aligned}
\mathbf{E}\widehat{\mathcal{O}}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}] &= \frac{\mathbf{E}\mathcal{O}[a\cdot p - \frac{1}{2}a\cdot q_- - \frac{1}{2}a\cdot q_+]\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]}{\mathbf{E}\mathcal{O}[a\cdot p - \frac{1}{2}a\cdot q_- - \frac{1}{2}a\cdot q_+]} \\
&= \frac{\mathbf{E}\mathcal{O}[\boldsymbol{\sigma} + a\cdot p - \frac{1}{2}a\cdot q_- - \frac{1}{2}a\cdot q_+, \boldsymbol{\sigma}_*]}{\mathbf{E}\mathcal{O}[a\cdot p - \frac{1}{2}a\cdot q_- - \frac{1}{2}a\cdot q_+]} \\
&= \frac{\mathbf{E}\mathcal{P}_{(b)}^{-1}\mathcal{C}_{(b)}[\boldsymbol{\sigma} + a\cdot p + (b - \frac{1}{2}a)\cdot q_- + (b - \frac{1}{2}a)\cdot q_+, \boldsymbol{\sigma}_*]}{\mathbf{E}\mathcal{P}_{(b)}^{-1}\mathcal{C}_{(b)}[a\cdot p + (b - \frac{1}{2}a)\cdot q_- + (b - \frac{1}{2}a)\cdot q_+]},
\end{aligned}$$

the remaining part follows from the definition of Coulomb gas correlation functions.

6.2 BPZ-Cardy equations

We now derive BPZ-Cardy equations in the dipolar case. Suppose $X = X_1(z_1)\cdots X_n(z_n)$ is the tensor product of fields X_j in the OPE family $\mathcal{F}_{(b)}$. For $\xi \in \mathbb{R}$, we denote

$$\widehat{\mathbf{E}}_\xi[X] = \mathbf{E}[e^{\odot\frac{1}{2}ia(\Phi^+_{(0)}(\xi,-1)+\Phi^+_{(0)}(\xi,1))}X].$$

Proposition 6.2.1. *If* $2a(a+b) = 1$, *then we have*

$$\widehat{\mathbf{E}}_\xi[\mathcal{L}_{v_\xi}X] = \frac{1}{2a^2}\Big(\frac{(1-\xi^2)^2}{2}\partial_\xi^2 - \xi(1-\xi^2)\partial_\xi\Big)\widehat{\mathbf{E}}_\xi[X], \quad v_\xi(z) := \frac{1-z^2}{2}\frac{1-\xi z}{\xi - z}, \tag{6.2.1}$$

where all fields are evaluated in the identity chart of $\mathbb{H}$ *and* $\partial_\xi = \partial + \bar{\partial}$.

Proof. In the $(\mathbb{H}, 0, -1, 1)$-uniformization, the rooted vertex field Ψ is evaluated at ξ as

$$\Psi(\xi) = (1-\xi^2)^{-h} e^{\odot \frac{1}{2} ia(\Phi^+_{(0)}(\xi,-1)+\Phi^+_{(0)}(\xi,1))},$$

where $h = \frac{1}{2}a^2 - ab$. For $\zeta \in \mathbb{H}$, let

$$R_\zeta \equiv R(\zeta; z_1, z_2, \cdots, z_n) = \mathbf{E}[(1-\zeta^2)^h \Psi(\zeta) X].$$

By Ward's equation (Proposition 4.3.1), $L_{-1}\Psi = \partial\Psi$, and level two degeneracy equation (Proposition 5.6.1) for the rooted vertex field Ψ, we have

$$\begin{aligned}
&\mathbf{E}[\Psi(\zeta)(\mathcal{L}^+_{v_\zeta} X + \mathcal{L}^-_{v_{\bar\zeta}} X)] \\
&= \frac{1}{2a^2}\frac{(1-\zeta^2)^2}{2}\mathbf{E}[(\partial^2\Psi)(\zeta)X] - \frac{3\zeta(1-\zeta^2)}{2}\mathbf{E}[(\partial\Psi)(\zeta)X] \\
&+ (\frac{3\zeta^2-1}{2}h + b^2 - \frac{a^2}{4})\mathbf{E}[\Psi(\zeta)X].
\end{aligned}$$

(We also use the fact that Ψ is a holomorphic field, and therefore $\mathcal{L}^-_v \Psi(\zeta) = 0$.) Due to the numerology $2a(a+b) = 1$, it is simplified that

$$\mathbf{E}[(1-\zeta^2)^h \Psi(\zeta)(\mathcal{L}^+_{v_\zeta} X + \mathcal{L}^-_{v_{\bar\zeta}} X)] = \frac{1}{2a^2}\Big(\frac{(1-\zeta^2)^2}{2}\partial^2 R_\zeta - \zeta(1-\zeta^2)\partial R_\zeta\Big),$$

where ∂ is the operator of differentiation with respect to the complex variable ζ. We now take the limits of both sides as $\zeta \to \xi$. Since ξ is real, the left-hand side converges to

$$\mathbf{E}[(1-\xi^2)^h \Psi(\xi)\mathcal{L}_{v_\xi} X] = \widehat{\mathbf{E}}_\xi[\mathcal{L}_{v_\xi} X].$$

On the other hand, since $\partial_\xi = \partial + \bar\partial$, and the rooted vertex field Ψ is holomorphic, ∂R_ζ, and $\partial^2 R_\zeta$ in the right-hand sides converge to $\partial_\xi R_\xi$ and $\partial^2_\xi R_\xi$, respectively. □

6.3 Dipolar SLE martingale-observables

It is convenient to describe dipolar SLEs in terms of the $(\mathbb{H}, -1, 1)$-uniformization. Let $\xi_t = (e^{\sqrt{\kappa}B_t} - 1)/(e^{\sqrt{\kappa}B_t} + 1)$ and let g_t be the dipolar SLE map from

(D_t, γ_t, Q) onto $(\mathbb{H}, \xi_t, \mathbb{R} \setminus [-1,1])$. Then g_t satisfies

$$\partial_t g_t(z) = -\frac{1-g_t^2(z)}{2} \frac{1-\xi_t g_t(z)}{\xi_t - g_t(z)}, \tag{6.3.1}$$

where $g_0 : (D, p, Q) \to (\mathbb{H}, 0, \mathbb{R} \setminus [-1,1])$ is the conformal map from D onto the upper half-plane $\mathbb{H}$. Let us restate Theorem 2.2.1 and present its proof. Recall that we defined $b \equiv b(\kappa) = \sqrt{\kappa/8} - \sqrt{2/\kappa}$ and now we define $a \equiv a(\kappa) = \sqrt{2/\kappa}$. Then the identity $2a(a+b) = 1$ holds for all κ.

Theorem 6.3.1. *If X_j's are Fock space fields in the OPE family $\mathcal{F}_{(b)}$, then a non-random field*

$$M(z_1, \cdots, z_n) = \widehat{\mathbf{E}}[X_1(z_1) \cdots X_n(z_n)]$$

is a martingale-observable for dipolar SLE_κ.

Proof. Conformal invariance allows us to represent the process

$$M_t(z_1, \cdots, z_n) \equiv M_{(D_t, \gamma_t, Q)}(z_1, \cdots, z_n)$$

as

$$M_t = m(\xi_t, t), \;\; m(\xi, t) = (R_\xi \,\|\, g_t^{-1}),$$

where $g_t : (D_t, \gamma_t, q_-, q_+) \to (\mathbb{H}, \xi_t, -1, 1)$ is the dipolar SLE map driven by ξ_t and

$$R_\xi(z_1, \cdots, z_n) = \widehat{\mathbf{E}}_\xi[X_1(z_1) \cdots X_n(z_n)].$$

Itô's formula can be applied to $m(\xi_t, t)$ since the function $m(\xi, t)$ is smooth in both ξ and t. Since the driving process $\xi_t = (e^{\sqrt{\kappa}B_t} - 1)/(e^{\sqrt{\kappa}B_t} + 1)$ satisfies

$$d\xi_t = \frac{\sqrt{\kappa}}{2}(1-\xi_t^2)\, dB_t - \frac{\kappa}{4}\, \xi_t (1-\xi_t^2)\, dt,$$

Itô's formula shows that M_t is a semi-martingale with the drift term

$$\frac{\kappa}{4}\Big(\frac{(1-\xi^2)^2}{2}\partial_\xi^2 - \xi(1-\xi^2)\partial_\xi\Big)\Big|_{\xi=\xi_t} m(\xi, t)\, dt + \frac{d}{ds}\Big|_{s=0}(R_\xi \,\|\, g_{t+s}^{-1})\, dt,$$

where $L_t = \partial_s\Big|_{s=0}(R_\xi \,\|\, g_{t+s}^{-1}) = \partial_s\Big|_{s=0}(R_\xi \,\|\, g_t^{-1} \circ f_{s,t}^{-1})$, and $f_{s,t} = g_{t+s} \circ g_t^{-1}$.

It follows from (6.3.1) that the time-dependent flow $f_{s,t}$ satisfies

$$\frac{d}{ds} f_{s,t}(\zeta) = -v_{\xi_{t+s}}(f_{s,t}(z)), \quad v_\xi(z) := \frac{1-z^2}{2}\frac{1-\xi z}{\xi - z}.$$

Thus $f_{s,t}$ can be approximated by $\mathrm{id} - s v_{\xi_t} + o(s)$ as $s \to 0$. Since X_j's depend smoothly on local charts,

$$L_t = -(\mathcal{L}_{v_{\xi_t}} R_{\xi_t} \parallel g_t^{-1}).$$

It follows from BPZ-Cardy equations that M_t is driftless. □

6.4 The restriction property

In this section we present CFT theoretic proof for the restriction property of the dipolar $\mathrm{SLE}_{8/3}$; the dipolar $\mathrm{SLE}_{8/3}$ path in $(\mathbb{H}, -1, 1)$ conditioned to avoid a fixed compact hull K with $\partial K \cap \mathbb{R} \subseteq (-1,1) \setminus \{0\}$ has the same distribution as the dipolar $\mathrm{SLE}_{8/3}$ path in $(\mathbb{H} \setminus K, -1, 1)$.

The restriction property of the chordal SLEs was described by Lawler, Schramm and Werner in [20]. We now briefly introduce the conformal restriction property and its applications. Let consider the upper half-plane $\mathbb{H}$ and two marked points 0 and ∞. The law of closed random subsets W of $\mathbb{H}$ is called a (chordal) restriction measure, if it satisfies the following properties:

- $\overline{W} \cap \mathbb{R} = \{0\}$, W is unbounded and $\mathbb{H} \setminus W$ has two connected components.
- For all simply connected subsets D of $\mathbb{H}$ such that $\mathbb{H} \setminus D$ is bounded and bounded away from the origin, the law of W conditioned on $W \subset D$ is equal to the law of $\psi(W)$, where ψ is a conformal map from $\mathbb{H}$ onto D that preserves the boundary points 0 and ∞.

It turns out that there exists only a one-parameter family $\mathbf{P}_\alpha$ of such probability measures, where α is a positive number, and that

$$\mathbf{P}_\alpha[W \subset D] = \psi'(0)^\alpha$$

where $\psi : D \mapsto \mathbb{H}$ is chosen in such a way that $\psi(z)/z \mapsto 1$ as $z \mapsto \infty$. The restriction property of the radial $\mathrm{SLE}_{8/3}$ was studied in [14]. Precisely, if γ is a radial $\mathrm{SLE}_{8/3}$ path in $\mathbb{D}$ from 1 to 0, and K is a compact set not containing 1, such that $\mathbb{D} \setminus K$ is simply connected and contains 0. Then,

$$\mathbf{P}[\gamma[0,\infty) \cap K = \emptyset] = |\psi_K'(0)|^{5/48} |\psi_K'(1)|^{5/8},$$

where ψ is a conformal map from $\mathbb{D} \setminus K$ onto $\mathbb{D}$ with $\psi_K(0) = 0$.

Let $\kappa \leq 4$. On the event $\gamma(0,\infty) \cap K = \emptyset$, we denote $\Omega_t = g_t(D_t \setminus K), \widetilde{\gamma} = \psi_K \circ \gamma$ and define a conformal map $h_t : \Omega_t \to \mathbb{H}$ by

$$h_t := \widetilde{g}_t \circ \psi_K \circ g_t^{-1},$$

where $\widetilde{g}_t$ is a dipolar Loewner map from $(\mathbb{H} \setminus \widetilde{\gamma}[0,t], -1, 1)$ onto $(\mathbb{H}, -1, 1)$ and ψ_K is the conformal transformation from $(\mathbb{H} \setminus K, -1, 1)$ onto $(\mathbb{H}, -1, 1)$ such that $\psi_K'(-1) = \psi_K'(1)$.

Let

$$M_t = (1 - \xi_t^2)^\lambda \mathbf{E}(\Psi^{\mathrm{eff}}_{\Omega_t}(\xi_t) \,\|\, \mathrm{id}_{\Omega_t}),$$

where $\Psi^{\mathrm{eff}} \equiv \Psi \mathcal{P}_{(b)}(q_\pm)$ is the effective boundary condition changing operator, see (2.4.3). Then

$$M_t = \Big(\frac{(1 - \xi_t^2) h_t'(\xi_t)}{1 - h_t(\xi_t)^2} \Big)^\lambda (h_t'(-1) h_t'(1))^\mu, \tag{6.4.1}$$

where

$$\lambda = h(\Psi^{\mathrm{eff}}) = \frac{6-\kappa}{2\kappa}, \quad \mu = h_-(\Psi^{\mathrm{eff}}) = h_+(\Psi^{\mathrm{eff}}) = \frac{a^2}{8} - \frac{b^2}{2} = \frac{(\kappa-2)(6-\kappa)}{16\kappa}.$$

Lemma 6.4.1. *The process M_t is a semi-martingale with the drift term*

$$\frac{c}{24}(1 - \xi_t^2)^2 \, S_{h_t}(\xi_t) \, M_t \, dt.$$

Proof. Let $F(z,t) := \mathbf{E}(\Psi^{\mathrm{eff}}_{\Omega_t}(z) \,\|\, \mathrm{id}_{\Omega_t})$. Then $M_t = (1 - \xi_t^2)^\lambda F(\xi_t, t)$. Application of Itô's formula to the smooth function F gives the drift term of dM_t/M_t :

$$\frac{\dot{F}(\xi_t,t)}{F(\xi_t,t)} - \frac{\kappa}{4}(1+2\lambda)(1-\xi_t^2)\xi_t \frac{F'(\xi_t,t)}{F(\xi_t,t)} + \frac{\kappa}{8}(1-\xi_t^2)^2 \frac{F''(\xi_t,t)}{F(\xi_t,t)} + \frac{\kappa}{2}(\lambda^2 + \frac{\lambda}{2})\xi_t^2 - \frac{\kappa}{4}\lambda.$$

We represent

$$\dot{F}(z,t) = \frac{d}{ds}\Big|_{s=0} \mathbf{E}(\Psi_{\mathbb{H}}^{\mathrm{eff}} \,\|\, h_{t+s}^{-1})(z) = \frac{d}{ds}\Big|_{s=0} \mathbf{E}(\Psi_{\mathbb{H}}^{\mathrm{eff}} \,\|\, h_t^{-1} \circ f_{s,t}^{-1})(z),$$

($f_{s,t} = h_{t+s} \circ h_t^{-1}$) in terms of Lie derivatives

$$\dot{F}(z,t) = \mathbf{E}(\mathcal{L}(v,\overline{\mathbb{H}})\Psi_{\mathbb{H}}^{\mathrm{eff}} \,\|\, h_t^{-1})(z), \qquad (v \,\|\, \mathrm{id}_{\mathbb{H}}) = \frac{d}{ds}\Big|_{s=0} f_{s,t} = \dot{h}_t \circ h_t^{-1}, \tag{6.4.2}$$

where the Lie derivative operator $\mathcal{L}(v,\overline{\mathbb{H}})$ applies to the points ± 1. To compute the vector field v, we apply the chain rule to $h_t = \widetilde{g}_t \circ \psi_K \circ g_t^{-1}$ and compute the capacity changes. Indeed,

$$\dot{h}_t(z) = -\Big(\frac{(1-\xi_t^2)h_t'(\xi_t)}{1-h_t(\xi_t)^2}\Big)^2 v_{h_t(\xi_t)}(h_t(z)) + h_t'(z) v_{\xi_t}(z),$$

where

$$v_\xi(z) := \frac{1-z^2}{2}\frac{1-\xi z}{\xi - z}.$$

Thus

$$(v \,\|\, \mathrm{id}_{\mathbb{H}})(\zeta) = -\Big(\frac{(1-\xi_t^2)h_t'(\xi_t)}{1-h_t(\xi_t)^2}\Big)^2 v_{h(\xi_t)}(\zeta) + h_t'(h_t^{-1}(\zeta))\, v_{\xi_t}(h_t^{-1}(\zeta)). \tag{6.4.3}$$

By (6.4.2) and (6.4.3), we have

$$\begin{aligned}\dot{F}(z,t) = &-\Big(\frac{(1-\xi_t^2)h_t'(\xi_t)}{1-h_t(\xi_t)^2}\Big)^2 h_t'(z)^\lambda\, \mathbf{E}(\mathcal{L}(v_{h_t(\xi_t)},\overline{\mathbb{H}})\Psi_{\mathbb{H}}^{\mathrm{eff}}(h_t(z)) \,\|\, \mathrm{id}_{\mathbb{H}}) \\ &+ \mathbf{E}(\mathcal{L}(v_{\xi_t},\Omega_t \setminus \{\pm 1\})\Psi_{\Omega_t}^{\mathrm{eff}} \,\|\, \mathrm{id}_{\Omega_t})(z) + 2\mu\, \mathbf{E}(\Psi_{\Omega_t}^{\mathrm{eff}} \,\|\, \mathrm{id}_{\Omega_t})(z),\end{aligned}$$

where the Lie derivative operator $\mathcal{L}(v_{\xi_t},\Omega_t \setminus \{\pm 1\})$ does not apply to the points ± 1. It follows from Ward's equation that

$$\begin{aligned}\dot{F}(z,t) = &-\frac{(1-\xi_t^2)^2}{2} h_t'(\xi_t)^2 h_t'(z)^\lambda \mathbf{E}(T_{\mathbb{H}}(h_t(\xi_t))\Psi_{\mathbb{H}}^{\mathrm{eff}}(h_t(z)) \,\|\, \mathrm{id}_{\mathbb{H}}) \\ &+ \mathbf{E}(\mathcal{L}(v_{\xi_t},\Omega_t \setminus \{\pm 1\})\Psi_{\Omega_t}^{\mathrm{eff}} \,\|\, \mathrm{id}_{\Omega_t})(z) + 2\mu\, \mathbf{E}(\Psi_{\Omega_t}^{\mathrm{eff}} \,\|\, \mathrm{id}_{\Omega_t})(z).\end{aligned}$$

By conformal invariance,

$$\begin{aligned}\dot{F}(z,t) &= -\frac{(1-\xi_t^2)^2}{2}\mathbf{E}(T_{\Omega_t}(\xi_t)\Psi^{\text{eff}}_{\Omega_t}(z) \,\|\, \text{id}_{\Omega_t}) \\ &+ \frac{c}{24}(1-\xi_t^2)^2 S_{h_t}(\xi_t)\mathbf{E}(\Psi^{\text{eff}}_{\Omega_t} \,\|\, \text{id}_{\Omega_t})(z) \\ &+ \mathbf{E}(\mathcal{L}(v_{\xi_t}, \Omega_t \setminus \{\pm 1\})\Psi^{\text{eff}}_{\Omega_t} \,\|\, \text{id}_{\Omega_t})(z) + 2\mu\, \mathbf{E}(\Psi^{\text{eff}}_{\Omega_t} \,\|\, \text{id}_{\Omega_t})(z).\end{aligned}$$

Using (4.3.1) and the fact that $L_{-1}\Psi = \partial\Psi, L_0\Psi = \lambda\Psi, L_1\Psi = 0$, we have

$$\begin{aligned}\frac{\dot{F}(\xi_t,t)}{F(\xi_t,t)} &= -\frac{(1-\xi_t^2)^2}{2}\frac{(\mathbf{E}\, T_{\Omega_t} * \Psi^{\text{eff}}_{\Omega_t}(\xi_t) \,\|\, \text{id})}{(\mathbf{E}\, \Psi^{\text{eff}}_{\Omega_t}(\xi_t) \,\|\, \text{id})} + \frac{3\xi_t(1-\xi_t^2)}{2}\frac{F'(\xi_t,t)}{F(\xi_t,t)} \\ &- (\frac{3\xi_t^2-1}{2}\lambda - 2\mu) + \frac{c}{24}(1-\xi_t^2)^2 S_{h_t}(\xi_t).\end{aligned}$$

Plugging the above equation into the drift term of dM_t, lemma now follows from the level two degeneracy equation for Ψ^{eff}. □

From now on, we use the $(\mathbb{S}, -\infty, \infty)$-uniformization. By abuse of notation, let $\xi_t = \sqrt{\kappa}B_t$ (cf. ξ_t in Section 6.3) and let g_t be the dipolar SLE map from (D_t, γ_t, Q) onto $(\mathbb{S}, \xi_t, \mathbb{R} + \pi i)$. As before, for a compact hull K with $K \cap (\mathbb{R} + \pi i) = \emptyset$, we denote $\Omega_t = g_t(D_t \setminus K), \widetilde{\gamma} = \psi_K \circ \gamma$ and define a conformal map $h_t : \Omega_t \to \mathbb{S}$ by

$$h_t := \widetilde{g}_t \circ \psi_K \circ g_t^{-1},$$

where $\widetilde{g}_t$ is a dipolar Loewner map from $(\mathbb{S} \setminus \widetilde{\gamma}[0,t], -\infty, \infty)$ onto $(\mathbb{S}, -\infty, \infty)$ and ψ_K is the conformal transformation from $(\mathbb{S} \setminus K, -\infty, \infty)$ onto $(\mathbb{S}, -\infty, \infty)$ such that

$$\lim_{z \to \pm\infty} \psi_K(z) - z = \pm\, \text{scap}(K).$$

Then the process (6.4.1) in the $(\mathbb{H}, -1, 1)$-uniformization becomes

$$M_t = (h_t'(\xi_t))^\lambda \, e^{-2\mu\, \text{scap}(K_t)}, \quad K_t = \overline{\mathbb{H} \setminus \Omega_t}$$

in the $(\mathbb{S}, -\infty, \infty)$-uniformization.

Proof of Theorem 2.4.1. By Lemma 6.4.1, the process

$$M_t = (h_t'(\xi_t))^\lambda \, e^{-2\mu\, \text{scap}(K_t)}$$

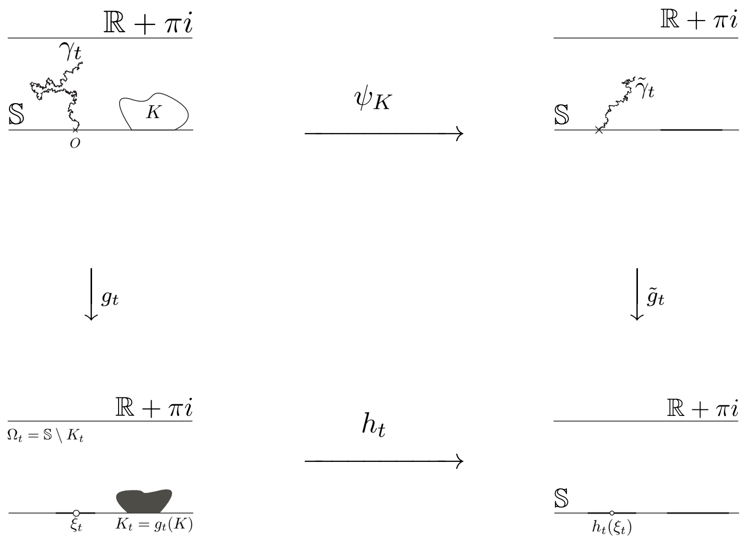

Figure 6.1: The conformal maps: $g_t, \tilde{g}_t, \psi_t, h_t$.

is a local martingale if $\kappa = 8/3$. We first claim that the process M_t is a bounded continuous martingale. Since $\operatorname{scap}(K) \geq 0$ for a compact hull K with $K \cap (\mathbb{R}+\pi i) = \emptyset$, it suffices to show that $(0 <)\psi_K'(x) \leq 1$ for $x \in \mathbb{R} \setminus K$. As in the chordal case, $\operatorname{Im}\psi_K(z) - \operatorname{Im} z$ is a bounded harmonic function with non-positive boundary values. Thus $\operatorname{Im}\psi_K(z) \leq \operatorname{Im} z$ for $z \in \mathbb{S} \setminus K$ and $\psi_K'(x) \leq 1$ for $x \in \mathbb{R} \setminus K$.

Let $T = \inf\{t \geq 0 : \gamma(0,t] \cap K \neq \emptyset\}$. It follows from the martingale convergence theorem that $\lim_{t\to T} M_t$ exists a.s. The proof that $\lim_{t\to T} M_t = 1_{T=\infty}$ a.s. is similar to that in the chordal case (see [20, Theorem 6.1]). We leave it as an exercise for the reader. By the optional stopping theorem,

$$\psi_K'(0)^\lambda e^{-2\mu \operatorname{scap}(K)} = M_0 = \mathbf{E}\, M_T = \mathbb{P}\{T = \infty\}.$$

This proves the theorem. $\square$

As an application of the restriction property of dipolar $\mathrm{SLE}_{8/3}$, we prove a dipolar version of Friedrich-Werner's formula. In [8, Proposition 1], Friedrich and Werner introduce a recursive formula which plays the role of the Ward identity in the conformal field theory. More precisely, let $B_n(x_1, \cdots x_n)$ be the normalized probabilities of the event that the chordal $\mathrm{SLE}_{8/3}$ hits all small slits rooted at $x_1, \cdots, x_n$. Then, B_n satisfies

$$\begin{aligned}B_{n+1}(x, x_1, \cdots, x_n) =& \frac{5/8}{x^2} B_n(x_1, \cdots, x_n) \\ & - \sum_{j=1}^{n} \{ (\frac{1}{x_j - x} + \frac{1}{x}) \partial_{x_j} - \frac{2}{(x_j - x)^2} \} B_n(x_1, \cdots, x_n).\end{aligned}$$

In fact, the correlation function $\widehat{\mathbf{E}}[T(x_1) \cdots T(x_n)]$ has exactly same recursive formula as above equation when $\kappa = 3/8$ i.e, $c = 0$ and $h_{1,2} = \frac{5}{8}$, see [12, Lecture 14.5].

We now prove Friedrich-Werner's formula in the dipolar case.

Proof of Theorem 2.4.2. Denote $\boldsymbol{x} = (x_1, \cdots, x_n)$, and

$$R(\xi; \boldsymbol{x}) = \mathbf{E} [\, \Psi^{\text{eff}}(\xi) \, T(x_1) \cdots T(x_n) \,].$$

The non-random field $R \equiv R(\xi; \boldsymbol{x})$ has the following properties:

(R1) it is a boundary differential of dimension $\lambda = 5/8$ with respect to ξ;

(R2) it is a boundary differential of dimension 2 with respect to x_j;

(R3) it is a boundary differential of dimension $\mu = 5/96$ with respect to $q_\pm$.

We apply Ward's equations to the function $R(\xi; \boldsymbol{x})$ so that we replace $T(x)$ in $R(\xi; x, \boldsymbol{x})$ by the Lie derivative operator:

$$R(\xi; x, \boldsymbol{x}) = (\mathcal{L}(v_x, \bar{\mathbb{S}}) + \mu) \, R(\xi; \boldsymbol{x}), \quad (\text{in } \mathrm{id}_{\bar{\mathbb{S}}}), \tag{6.4.4}$$

where $v_x(\zeta) = \frac{1}{2} \coth_2(x - \zeta)$ and the Lie derivative operator $\mathcal{L}(v_x, \bar{\mathbb{S}})$ do not apply to the points $\pm\infty$.

Let

$$U(\boldsymbol{x}) = \lim_{\varepsilon \to 0} \varepsilon^{-2n} \mathbb{P}(\text{SLE}_{8/3} \text{ hits all slits } [x_j, x_j + i\varepsilon\sqrt{2}])$$

(if the limit exists). We define the non-random field $T(\xi; \boldsymbol{x})$ as follows:

- it satisfies the transformation laws (R1) – (R3);
- $(T(\xi; x_1, \cdots, x_n) \, \| \, \mathrm{id}_{\bar{\mathbb{S}}}) = U(x_1 - \xi, \cdots, x_n - \xi)$.

We now claim that the limit $U(x, \boldsymbol{x})$ exists under the assumption of existence of the limit $U(\boldsymbol{x})$ and that

$$T(0; x, \boldsymbol{x}) = (\mathcal{L}(v_x, \bar{\mathbb{S}}) + \mu)\, T(0; \boldsymbol{x}). \tag{6.4.5}$$

The non-random fields $T(0; \cdot)$ and $R(0; \cdot)$ satisfy the same recursive equation (see (6.4.4) and (6.4.5)). Thus $T(0; \cdot) = R(0; \cdot)$ for all $n \geq 1$ since $T(0; \cdot) = R(0; \cdot) = 1$ for $n = 0$. Therefore, we have $U(\boldsymbol{x}) = R(0; \boldsymbol{x})$.

To verify this claim, we write $\mathbb{P}(\boldsymbol{x})$ for the probability that dipolar $\mathrm{SLE}_{8/3}$ path hits all segments $[x_j, x_j + i\varepsilon\sqrt{2}]$ and $\mathbb{P}(\boldsymbol{x} \,|\, \neg x)$ for the same probability conditioned on the event that the path avoids $[x, x + i\varepsilon\sqrt{2}]$. By the induction hypothesis,

$$\mathbb{P}(\boldsymbol{x}) \approx \varepsilon^{2n}\, T(0; \boldsymbol{x}). \tag{6.4.6}$$

On the other hand, it follows from the restriction property of dipolar $\mathrm{SLE}_{8/3}$ that

$$1 - \mathbb{P}(x) = (\psi'(0))^{\lambda}\, (e^{-\mathrm{scap}([x, x+i\varepsilon\sqrt{2}])})^{2\mu} \tag{6.4.7}$$

and

$$\mathbb{P}(\boldsymbol{x} \,|\, \neg x) \approx \varepsilon^{2n}\, T(\psi(0); \psi(x_1), \cdots, \psi(x_n)) \prod_{j=1}^{n} \psi'(x_j)^2, \tag{6.4.8}$$

where ψ is a slit map from $(\mathbb{S} \setminus [x, x + i\varepsilon\sqrt{2}], 0, \pm\infty)$ onto $(\mathbb{S}, 0, \pm\infty)$.

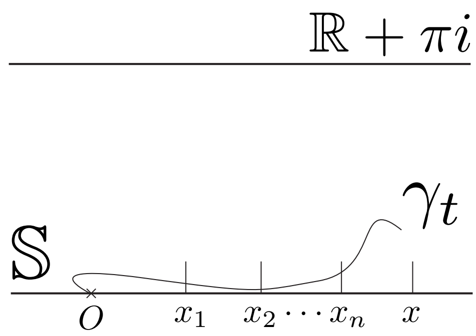

Figure 6.2: Example of the event $(\boldsymbol{x} \,|\, \neg x)$.

This ψ satisfies

$$\psi(z) = \varphi_t(z - x) - \varphi_t(-x), \quad \cosh_2 \varphi_t(z) = e^{\frac{1}{2}t} \cosh_2 z, \quad e^t = 1 + \tan^2(\varepsilon/\sqrt{2}),$$

where $\cosh_2 z = \cosh(\frac{1}{2}z)$. By (6.4.6) – (6.4.8), we approximate $\varepsilon^{-2n}\mathbb{P}(x, \boldsymbol{x})$ by

$$T(0;\boldsymbol{x}) - \psi'(0)^{\lambda}(e^{-\mathrm{scap}([x,x+i\varepsilon\sqrt{2}])})^{2\mu}\, T(\psi(0);\psi(x_1),\cdots,\psi(x_n))\prod_{j=1}^{n}\psi'(x_j)^2.$$

Thus the limit $U(x, \boldsymbol{x})$ exists. Since $\mathrm{scap}([x, x+i\varepsilon\sqrt{2}]) = \log(1+\tan_2^2\sqrt{2}\varepsilon) \approx \frac{1}{2}\varepsilon^2$, we have

$$T(0; x, \boldsymbol{x}) = (\mathcal{L}(v_x, \bar{\mathbb{S}}) + \mu)\, T(0;\boldsymbol{x}).$$

$\square$

Chapter 7

Vertex observables

We expand our OPE family $\mathcal{F}_{(b)}$ of $\Phi_{(b)}$ by considering the rooted multi-vertex fields with the neutrality condition. In this chapter, we extend Theorem 2.2.1 to this expanded family. As examples of screening of rooted vertex observables, we discuss Cardy-Zhan observables that describe the probability for a point to be to the left (right) of the dipolar SLE path and the probability for a point to be swallowed by the dipolar SLE hull.

7.1 Rooted Multi-vertex fields

We apply the rooting rules (in section 5.4) to the multi-vertex fields $\widehat{\mathcal{O}}^{(\boldsymbol{\sigma},\boldsymbol{\sigma_*})}$ $\widehat{\mathcal{O}}^{(\sigma_-)}\widehat{\mathcal{O}}^{(\sigma_+)}$ and arrive to the definition of rooted multi-vertex fields

$$\widehat{\mathcal{O}}[\boldsymbol{\sigma},\boldsymbol{\sigma_*};\sigma_-,\sigma_+] = \widehat{M}[\boldsymbol{\sigma},\boldsymbol{\sigma_*};\sigma_-,\sigma_+]\, e^{\odot i\Phi[\boldsymbol{\sigma},\boldsymbol{\sigma_*};\sigma_-,\sigma_+]},$$

where $\widehat{M}[\boldsymbol{\sigma},\boldsymbol{\sigma_*};\sigma_-,\sigma_+] = \mathbf{E}\widehat{\mathcal{O}}[\boldsymbol{\sigma},\boldsymbol{\sigma_*};\sigma_-,\sigma_+] = (w_-')^{\widehat{h}_-}(w_+')^{\widehat{h}_+}\prod \widehat{M}_j \prod I_{j,k}$. The interaction term $I_{j,k}$ is same as (5.2.2) and

$$\begin{aligned}\widehat{M}_j \equiv \widehat{M}^{(\sigma_j,\sigma_{*j})}(z_j) &= (w_j')^{h_j}(\overline{w_j'})^{h_{*j}}(w_j-\bar{w}_j)^{\sigma_j\sigma_{*j}}\\ &\times w_j^{\sigma_j a}\bar{w}_j^{\sigma_{*j}a}(1-w_j)^{\widehat{\nu}_+}(1+w_j)^{\widehat{\nu}_-}(1-\bar{w}_j)^{\widehat{\nu}_{*+}}(1+\bar{w}_j)^{\widehat{\nu}_{*-}}\end{aligned}$$

with the exponents $\widehat{\nu}_\pm = \sigma_j(b-\frac{1}{2}a+\sigma_\pm), \widehat{\nu}_{*\pm} = \sigma_{*j}(b-\frac{1}{2}a+\sigma_\pm)$. The dimensions $[\boldsymbol{h},\boldsymbol{h}_*;\widehat{h}_-,\widehat{h}_+]$ of rooted multi-vertex fields $\widehat{\mathcal{O}}[\boldsymbol{\sigma},\boldsymbol{\sigma_*};\sigma_-,\sigma_+]$ are given by

$$h_j = \frac{\sigma_j^2}{2} - \sigma_j b, \qquad h_{*j} = \frac{\sigma_{*j}^2}{2} - \sigma_{*j} b, \qquad \widehat{h}_\pm = \frac{\sigma_\pm^2}{2} - \frac{\sigma_\pm a}{2}.$$

Alternatively, one can define $\widehat{\mathcal{O}}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+]$ by the action of boundary condition changing operator $\mathcal{X} \mapsto \widehat{\mathcal{X}}$ on $\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+]$. Indeed, the boundary condition changing operator $\mathcal{X} \mapsto \widehat{\mathcal{X}}$ can be extended to formal fields/functionals by the formula

$$\Phi^+_{(0)} \mapsto \Phi^+_{(0)} - \frac{ia}{2} \log \frac{w^2}{1-w^2}, \qquad \Phi^+_{(0)}(q_\pm) \mapsto \Phi^+_{(0)}(q_\pm) + \frac{ia}{2} \log w'_\pm$$

and the property that it commutes with complex conjugation. Note that the interaction terms are preserved under the boundary condition changing operator. For the rooted multi-vertex field $\mathcal{O} \equiv \mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+]$,

$$\widehat{\mathcal{O}} = (w'_-)^{-\frac{1}{2}\sigma_- a}(w'_+)^{-\frac{1}{2}\sigma_+ a} \prod w_j^{\sigma_j a}(1-w_j^2)^{-\frac{1}{2}\sigma_j a} \bar{w}_j^{\sigma_{*j} a}(1-\bar{w}_j^2)^{-\frac{1}{2}\sigma_{*j} a}\, \mathcal{O}.$$

Thus two definitions coincide.

For rooted multi vertex fields $\mathcal{O} \equiv \mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+]$ with the neutrality condition, let us denote

$$\widehat{\mathbf{E}}\,[\mathcal{O}] := \frac{\mathbf{E}[\Psi(p) \star \mathcal{O}]}{\mathbf{E}\Psi(p)}, \qquad \widehat{\mathbf{E}}_\zeta[\mathcal{O}] := \frac{\mathbf{E}[\Psi(\zeta) \star \mathcal{O}]}{\mathbf{E}\Psi(\zeta)}, \quad (\zeta \in \bar{D} \setminus \{q_\pm\}).$$

Then (6.1.1) can be extended to the rooted multi-vertex fields:

$$\widehat{\mathbf{E}}\,[\mathcal{O}] = \mathbf{E}\,[\widehat{\mathcal{O}}]. \tag{7.1.1}$$

Recall that $\Psi(p)$ is given by $\mathcal{O}[a \cdot p - \frac{1}{2}a \cdot q_- - \frac{1}{2}a \cdot q_+]$. Hence, if $[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+]$ satisfies the neutrality condition (NC_0), then we have

$$\begin{aligned} &\mathbf{E}\Psi(p) \star \mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+] = \mathbf{E}\mathcal{O}[\boldsymbol{\sigma} + a \cdot p, \boldsymbol{\sigma}_*; -\frac{1}{2}a + \sigma_-, -\frac{1}{2}a + \sigma_+] \\ &= \mathcal{P}_{(b)}^{-1}\mathcal{C}_{(b)}[\boldsymbol{\sigma} + a \cdot p + (b - \frac{1}{2}a + \sigma_-) \cdot q_- + (b - \frac{1}{2}a + \sigma_+) \cdot q_+, \boldsymbol{\sigma}_*]. \end{aligned}$$

Therefore, the correlation function $\widehat{M}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+]$ is given by

$$\widehat{M}[\boldsymbol{\sigma}, \boldsymbol{\sigma_*}; \sigma_-, \sigma_+] = \frac{\mathcal{C}[\boldsymbol{\sigma} + a \cdot p + (b - \frac{1}{2}a + \sigma_-) \cdot q_- + (b - \frac{1}{2}a + \sigma_+) \cdot q_+, \boldsymbol{\sigma}_*]}{\mathcal{C}[a \cdot p + (b - \frac{1}{2}a) \cdot q_- + (b - \frac{1}{2}a) \cdot q_+, 0]}.$$

For example, we can compute the boundary dimensions at $q_\pm$ as the following way;

$$h(b-\frac{1}{2}a+\sigma_\pm)-h(b-\frac{1}{2}a)=\frac{\sigma_\pm^2}{2}-\frac{\sigma_\pm a}{2},$$

where $h(x)=x^2/2-bx$. Also, the term $(1-w_j)^{\widehat{\nu}_+}$ comes from the Coulomb interaction between $\sigma_j\cdot z_j$ and $(b-\frac{1}{2}a+\sigma_+)\cdot q_+$.

Proposition 6.2.1 (BPZ-Cardy equations) extends to the rooted multi-vertex fields.

Proposition 7.1.1. *Suppose that the parameters a and b are related as $2a(a+b)=1$. Then for rooted multi-vertex fields $\mathcal{O}\equiv\mathcal{O}[\boldsymbol{\sigma},\boldsymbol{\sigma_*};\sigma_-,\sigma_+]$ with the neutrality condition,*

$$\widehat{\mathbf{E}}_\xi[\mathcal{L}_{v_\xi}\mathcal{O}]+(\widehat{h}_-+\widehat{h}_+)\widehat{\mathbf{E}}_\xi[\mathcal{O}]=\frac{1}{2a^2}\Big(\frac{(1-\xi^2)^2}{2}\partial_\xi^2-\xi(1-\xi^2)\partial_\xi\Big)\widehat{\mathbf{E}}_\xi[\mathcal{O}],\quad(7.1.2)$$

where all fields are evaluated in the identity chart of the upper half-plane and $\partial_\xi=\partial+\bar\partial$. The vector field v_ξ is given by

$$v_\xi(z):=\frac{1-z^2}{2}\frac{1-\xi z}{\xi-z}.$$

Proof. For $\zeta\in\overline{\mathbb{H}}\setminus\{\pm1\}$, let us denote

$$R_\zeta\equiv R(\zeta,z_1,\cdots,z_n)=\mathbf{E}[(1-\zeta^2)^h\Psi(\zeta)\star\mathcal{O}],$$

where $h=\frac{1}{2}a^2-ab$. Then it follows from Ward's equation (Proposition 5.5.3), $L_{-1}\Psi=\partial\Psi$, and the level two degeneracy equation for Ψ (Proposition 5.6.1) that

$$\begin{aligned}&\mathbf{E}[\Psi\star(\mathcal{L}^+_{v_\zeta}\mathcal{O}+\mathcal{L}^-_{v_{\bar\zeta}}\mathcal{O})]\\&=\frac{1}{2a^2}\frac{(1-\zeta^2)^2}{2}\mathbf{E}[(\partial^2\Psi)\star\mathcal{O}]-\frac{3\zeta(1-\zeta^2)}{2}\mathbf{E}[(\partial\Psi)\star\mathcal{O}]\\&+(\frac{3\zeta^2-1}{2}h+b^2-h_-(\Psi\star\mathcal{O})-h_+(\Psi\star\mathcal{O}))\mathbf{E}[\Psi\star\mathcal{O}],\end{aligned}$$

where $h_\pm(\Psi\star\mathcal{O})$ is the dimension of boundary differential $\Psi\star\mathcal{O}$ with respect to $q_\pm$. (We also use the holomorphicity of Ψ, and therefore $\mathcal{L}_v^-\Psi(\zeta)=0$.) By the numerology $2a(a+b)=1$ and the relation $h_\pm(\Psi\star\mathcal{O})=\frac{1}{8}a^2+\widehat{h}_\pm$, we have

$$\mathbf{E}[(1-\zeta^2)^h \Psi \star (\mathcal{L}^+_{v_\zeta}\mathcal{O} + \mathcal{L}^-_{v_{\bar\zeta}}\mathcal{O})] + (\widehat{h}_- + \widehat{h}_+)\mathbf{E}[(1-\zeta^2)^h \Psi \star \mathcal{O}]$$
$$= \frac{1}{2a^2}\Big(\frac{(1-\zeta^2)^2}{2}\partial^2 - \zeta(1-\zeta^2)\partial\Big) R_\zeta,$$

where ∂ is the operator of differentiation with respect to the complex variable ζ. Taking the limits of both sides as $\zeta \to \xi$, we obtain BPZ-Cardy equations. □

Now we prove Theorem 2.3.1.

Proof of Theorem 2.3.1. Denote

$$R_\xi = \widehat{\mathbf{E}}_\xi[\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*; \sigma_-, \sigma_+]].$$

By conformal invariance, the process $M_t \equiv M_{(D_t, \gamma_t, Q)}$ is represented by

$$M_t = m(\xi_t, t), \ m(\xi, t) = (R_\xi \,\|\, g_t^{-1}),$$

where g_t is the dipolar SLE map from $(D_t, \gamma_t, q_\pm)$ onto $(\mathbb{H}, \xi_t, \pm 1)$. Since $g_t'(q_\pm) = e^{-t}$, the drift term of dM_t is equal to

$$\frac{1}{2a^2}\Big(\frac{(1-\xi^2)^2}{2}\partial_\xi^2 - \xi(1-\xi^2)\partial_\xi\Big)\Big|_{\xi=\xi_t} m(\xi,t) - (\mathcal{L}_{v_{\xi_t}} R_t \,\|\, g_t^{-1}) - (\widehat{h}_- + \widehat{h}_+) M_t,$$

where the Lie derivative operator $\mathcal{L}_{v_{\xi_t}}$ does not apply the marked boundary points $q_\pm$. It follows from Proposition 7.1.1 (BPZ-Cardy equations) that dM_t is driftless. □

7.2 Cardy-Zhan observables

Let $\kappa > 4, z \in D$. We consider the following geometric observables

$$N(z) = \mathbb{P}(\tau_z < \infty), \quad \mathbb{P}(z \text{ is to the left of } \gamma), \quad \mathbb{P}(z \text{ is to the right of } \gamma).$$

They are real-valued with all conformal dimensions zero. There is no such vertex observable except for the constant field. However, the derivative

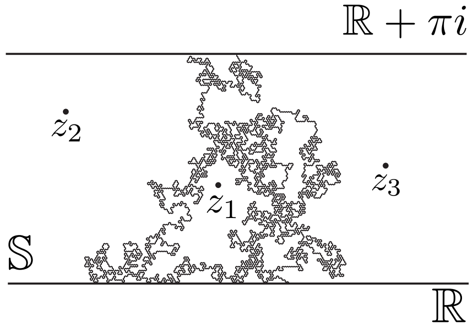

Figure 7.1: Example of the events: $\tau_{z_1} < \infty$, z_2 is to the left of γ and z_3 is to the right of γ.

∂N can be identified as a vertex field with conformal dimensions

$$\lambda_z = 1, \quad \lambda_{z*} = \lambda_{q+} = \lambda_{q-} = 0.$$

Indeed, by dimension calculus, a vertex field $\widehat{\mathcal{O}}[-2a \cdot z; a, a]$ satisfies the above requirements. Let M be a martingale observable with all conformal dimensions zero such that

$$\partial M(z) = \mathbf{E}\widehat{\mathcal{O}}[-2a \cdot z; a, a] = w' \sinh^{-4/\kappa}(\frac{w}{2}) \quad \text{up to multiplicative constant,}$$

where w is the conformal map from $(D, p, q_\pm)$ onto $(\mathbb{S}, 0, \pm\infty)$. Let us choose M satisfying the normalization $M(-\infty) = 0$, $M(\infty) = 1$, and $\operatorname{Im} M(0) > 0$. It follows from Schwarz-Christoffel formula that M is the conformal transformation from $(\mathbb{S}, 0, \pm\infty)$ onto the isosceles triangle with angles $2\pi/\kappa$ at $M(-\infty) = 0$, $M(\infty) = 1$. Applying the optional stopping theorem and using the fact that

$$\begin{cases} M_{\tau_z} = M(0) & \text{if } \tau_z < \infty, \\ M_{\tau_z} = M(-\infty) = 0 & \text{if } z \text{ is to the left of } \gamma, \\ M_{\tau_z} = M(\infty) = 1 & \text{if } z \text{ is to the right of } \gamma, \end{cases}$$

we justify Cardy-Zhan formulas (see [25, Corollary 2.3.1])

$$\mathbb{P}(\tau_z < \infty) = \frac{\operatorname{Im} M(z)}{\operatorname{Im} M(0)}, \quad \mathbb{P}(z \text{ is to the right of } \gamma) = \operatorname{Re} M(z) - \frac{1}{2}\frac{\operatorname{Im} M(z)}{\operatorname{Im} M(0)}.$$

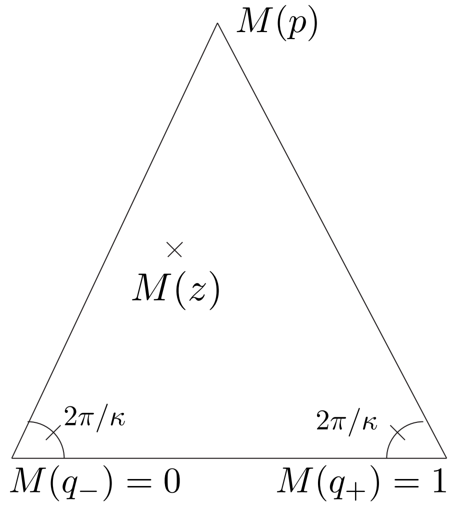

Figure 7.2: The image of M.

Remark. The Cardy observables for chordal SLE was studied in [12, Lecture 15.4] using similar argument. Let $\kappa > 4$, $z \in D$, $\eta \in \partial D$ where p, η, q are positively oriented. Also, let τ_z be the swallowed time by the SLE_κ from p to q. Three geometric observables

$$N(z,\eta) = \mathbf{P}(\tau_z < \tau_\eta), \quad \mathbf{P}(\tau_z = \tau_\eta), \quad \mathbf{P}(\tau_z > \tau_\eta)$$

are real-valued with all conformal dimensions zero. There is no such vertex observable except for the constant field. Hence, as above, the derivative $\partial_z N(z,\eta)$ can be identified a multi-vertex field with conformal dimensions

$$\lambda_z = 1, \quad \lambda_{*z} = \lambda_\eta = \lambda_q = 0.$$

Let M be a chordal SLE martingale observable such that

$$\partial M = \text{const.}\mathbf{E}\mathcal{O}^{(-2a)}(z) \star \mathcal{O}^{(2b)}(\eta) = \text{const.} w_z^{-\frac{4}{\kappa}} w_\eta^{1-\frac{4}{\kappa}} (w_z - w_\eta)^{\frac{8}{\kappa}-2} w_z',$$

where w is a conformal map from (D, p, q) onto $(\mathbb{H}, 0, \infty)$. We take

$$M(z) = \frac{\int_p^z \mathbf{E}\mathcal{O}^{(-2a)}(\zeta) \star \mathcal{O}^{(2b)}(\eta) d\zeta}{\int_p^q \mathbf{E}\mathcal{O}^{(-2a)}(\zeta) \star \mathcal{O}^{(2b)}(\eta) d\zeta},$$

the integral converges because $\kappa > 4$. By Schwarz-Christoffel formula, M is a conformal map onto the isosceles triangle with angles $(1-4/\kappa)\pi$ at $M(p) = 0$ and $M(q) = 1$, and angle $(8/\kappa - 1)\pi$ at $M(\eta)$ with $\mathrm{Re}M(\eta) = 1/2$. Applying the optional stopping theorem and the fact that

$$\begin{cases} M_{\tau_z} = M(p) = 0 & \text{if } \tau_z < \tau_\eta, \\ M_{\tau_z} = M(q) = 1 & \text{if } \tau_z > \tau_\eta, \\ M_{\tau_z} = M(\eta) & \text{if } \tau_z = \tau_\eta, \end{cases}$$

we justify Cardy formulas

$$\mathbf{P}(\tau_z > \tau_\eta) = \mathrm{Re}M(z) - \frac{\mathrm{Im}M(z)}{2\mathrm{Im}M(\eta)}, \quad \mathbf{P}(\tau_z = \tau_\eta) = \frac{\mathrm{Im}M(z)}{\mathrm{Im}M(\eta)}.$$

Remark. The chordal SLE (the radial SLE, respectively) ends at the marked boundary point (the marked interior point, respectively) q almost surely. However, dipolar SLE_κ $(\kappa > 0)$ path has a random endpoint $\gamma(\infty)$, but almost surely $\gamma(\infty) \in Q$ where Q is the boundary arc from q_- to q_+. If $z = x + \pi i \in \mathbb{R} + \pi i$, then Cardy-Zhan observables

$$M(x + \pi i) = \frac{\displaystyle\int_{-\infty}^{x} \cosh^{-4/\kappa}(\frac{\xi}{2})\, d\xi}{\displaystyle\int_{-\infty}^{\infty} \cosh^{-4/\kappa}(\frac{\xi}{2})\, d\xi}$$

are the cumulative distribution functions of the endpoint $\gamma(\infty)$ of dipolar SLE_κ $(\kappa > 0)$ path, see [25, Theorem 2.3.2].

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국문초록

이 학위 논문에서는 디리클렛 경계 조건을 갖는 가우시안 자유장(Gaussian free field)의 중심전하변화에 따른 쌍극자 등각장론에 관해 연구하고, 경계 조건 변환자 하에서 특정한 모임에 속하는 장들의 상관함수가 쌍극자 슈람-뢰브너 전개의 마팅게일 관측자임을 증명하였다.
쌍극자 등각장론을 도입하기 위해서 먼저 chordal 등각장론에서 다루는 정의와 기본 성질들을 기술하였다. 쌍극자 등각장론을 이용하여 SLE(8/3)의 제한 성질(restriction property) 및 Friedrich-Werner의 공식과 Cardy-Zhan 관측자에 관해 서술하였다.

주요어휘: 쌍극자 등각장론, 쌍극자 슈람-뢰브너 전개, 마팅게일-관측자
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